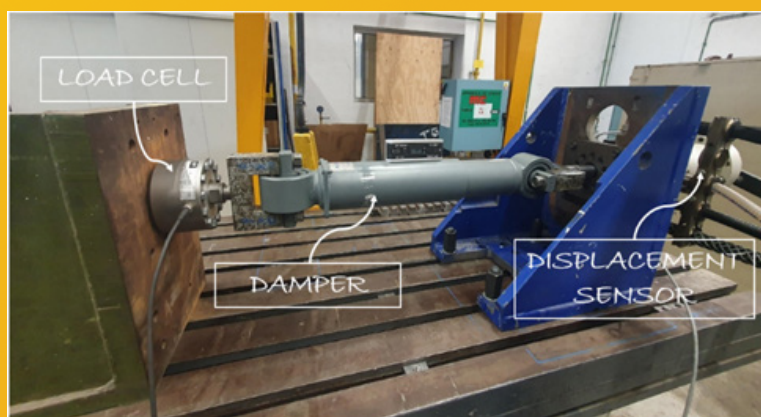


# Model Validation and Uncertainty Quantification, Vol. 3

Roland Platz  
Garrison Flynn  
Kyle Neal  
Scott Ouellette



Proceedings of the 43rd IMAC, A Conference and  
Exposition on Structural Dynamics 2025



# **Conference Proceedings of the Society for Experimental Mechanics Series**

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Roland Platz · Garrison Flynn · Kyle Neal · Scott Ouellette  
Editors

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# Preface

*Model Validation and Uncertainty Quantification* represents one of twelve volumes of technical papers presented at the 43<sup>rd</sup> IMAC, A Conference and Exposition on Structural Dynamics, organized by the Society for Experimental Mechanics, and held February 10-13, 2025. The full proceedings also include volumes on Nonlinear Structures & Systems; Dynamics of Civil Structures; Dynamic Substructuring & Transfer Path Analysis; Special Topics in Structural Dynamics & Experimental Techniques; Computer Vision & Laser Vibrometry; Dynamic Environments Testing; Sensors & Instrumentation and Aircraft/Aerospace Testing Techniques; Topics in Modal Analysis & Parameter Identification I&II; Data Science in Engineering; and Structural Health Monitoring & Machine Learning.

Each collection presents early findings from experimental and computational investigations on an important area within Structural Dynamics. Model Validation and Uncertainty Quantification (MVUQ) is one of these areas.

Modeling and simulation are routinely implemented to predict the behavior of complex dynamical systems. These tools powerfully unite theoretical foundations, numerical models and experimental data which include associated uncertainties and errors. The field of MVUQ research entails the development of methods and metrics to test model prediction accuracy and robustness while considering all relevant sources of uncertainties and errors through systematic comparisons against experimental observations.

The MVUQ Technical Division at the Society for Experimental Mechanics would like to thank the authors, presenters, session organizers, and session chairs for their participation and effort in this track.

Editor:

Roland Platz—*Deggendorf Institute of Technology*, Weißenburg, Bavaria, Germany; Garrison Flynn—*Los Alamos National Laboratory*, Los Alamos, NM, USA; Kyle Neal—*Sandia National Laboratories*, Albuquerque, NM, USA; Scott Ouellette—*Los Alamos National Laboratory*, Los Alamos, NM, USA.



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# Chapter 1

## Acoustic Characterization of Solid Periodic Ring Structures in Woodwind Musical Instruments

Alessandro Annessi, Alessia Caputo, Milena Martarelli, Daniele Eugenio Lucchetta, Davide Mencarelli, and Paolo Castellini

**Abstract** Woodwind musical instruments are a very interesting example of a vibro-acoustic device. They combine vibration, fluid dynamics and acoustics to create sound. While the basic concept of the instrument is, of course, well established, it is to this day possible to work on certain components for the purpose of improving, simply changing, certain characteristics of the instrument. Moreover, the interaction with the musician, who is in fact not only a performer but an integral part of the instrument, makes this analysis beyond challenging. In this paper, some innovative solutions will be analyzed by means of acoustic analysis with microphones and laser Doppler interferometry. The results obtained will be compared with numerical simulations in order to understand their behavior in depth.

**Keywords** Laser Doppler Vibrometry · Numerical Simulation · Periodic Structures · Musical Instruments · Acoustics

### Introduction

Woodwind musical instruments offer a fascinating interplay of vibrational, fluid-dynamic, and acoustic behaviors that contribute to sound production. Despite a long history, there remains significant potential for research and innovation in improving and altering specific characteristics of these instruments. The interaction between the musician and the instrument further complicates this dynamic system, making it a rich field for both experimental and numerical study. In a wind instrument, regardless of the reed generation configuration, the motion of the vibrating reed valve is controlled by the pressure difference between the inlet and the outlet of the valve. The vibration frequency depends partly on the natural frequency of the reed and partly on the resonances of the air column of the instrument to which it is connected. The pressure difference across the valve is usually large enough that Bernoulli's equation can be applied. For a woodwind reed, the volume flow can then be expressed as [1]:

$$U = \alpha(p_0 - p)^{\frac{1}{2}} - \beta(p_0 - p)^{\frac{3}{2}} \quad (1)$$

where  $\alpha$  and  $\beta$  are positive constants. This equation indicates that as the blowing pressure increases, the flow  $U$  rises from zero to a maximum value. Beyond this point, further increase in pressure causes the steady flow to decrease back to zero. Single reed instruments, such as the saxophone or clarinet, operate in the decreasing part of the curve where the conductance is negative and the reed acts as an acoustic generator. Starting from equation (1), it is straightforward to derive the pressure value  $p_c$  corresponding to the reed closing position (zero flow condition). The pressure corresponding to the maximum flow

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Alessia Caputo · Milena Martarelli · Paolo Castellini

Department of Industrial Engineering and Mathematical Sciences, Polytechnic University of Marche, Ancona, Italy  
e-mail: [a.caputo@pm.univpm.it](mailto:a.caputo@pm.univpm.it); [m.martarelli@staff.univpm.it](mailto:m.martarelli@staff.univpm.it); [p.castellini@staff.univpm.it](mailto:p.castellini@staff.univpm.it)

Alessandro Annessi

Department of Agricultural, Food and Environmental Sciences, Polytechnic University of Marche, Ancona, Italy  
e-mail: [a.annessi@staff.univpm.it](mailto:a.annessi@staff.univpm.it)

Daniele Eugenio Lucchetta

Optoacoustic Lab, SIMAU Department, Polytechnic University of Marche, Ancona, Italy  
e-mail: [d.lucchetta@staff.univpm.it](mailto:d.lucchetta@staff.univpm.it)

Davide Mencarelli

Department of Information Engineering, Polytechnic University of Marche, Ancona, Italy  
e-mail: [d.mencarelli@staff.univpm.it](mailto:d.mencarelli@staff.univpm.it)

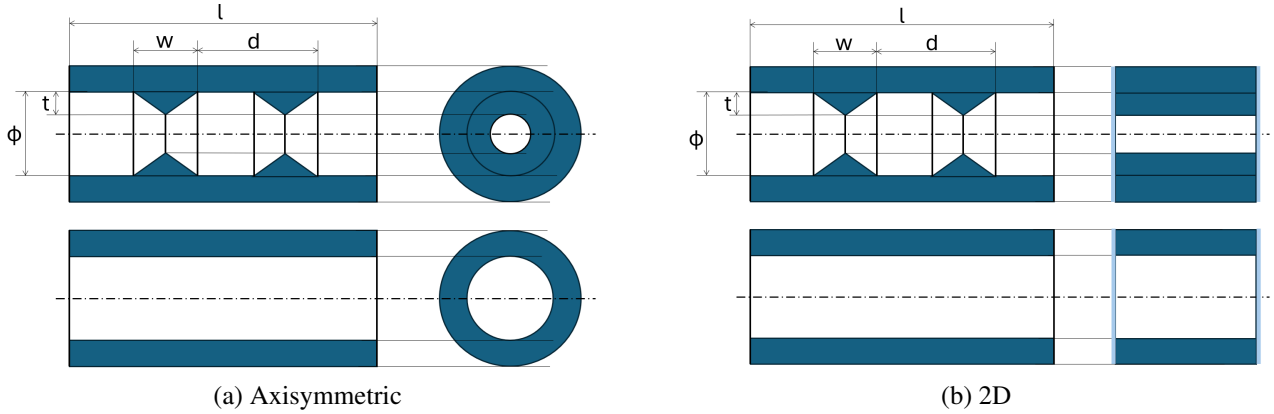
value is  $p_a = \frac{1}{3}p_c$ , and the player will use a pressure value between these points. Our goal is to determine the amplitude and shape of pressure and flow waveform fields under stationary conditions. Although the ideal lossless condition is a simplification, it serves as the best starting point for understanding the behavior of any single reed woodwind instrument, even when additional components that alter its acoustic properties are introduced. This study aims to introduce these additional components and examine their effects on the emitted sound spectra using an innovative experimental approach. The idea is to introduce solid periodic ring structures inside a small cylindrical volume to create sonic stop bands for selected frequencies. As a first step, before any practical application, we explored the possibility of attenuating the highest harmonic content of the input signal. In a clarinet, the highest note reaches 2 kHz, so our device should cut frequencies above 4 kHz to produce a mellow and pleasant sound, a strong requirement of classical musicians and the main motivation for our work. Recent advancements in 3D printing technologies have made it possible to fabricate periodic structures within the instrument's body. We tested several configurations through numerous simulations until we achieved promising results. Given the limited space inside the clarinet body, only a few ring element repetition (two or three) can be used. Limiting our choice to the simplest case of two elements, we found that rings with a triangular cross-section were more effective than other shapes. Varying a few geometrical parameters, such as the height of the ring, resulted in a significant change in the attenuation response. To accurately measure and analyze the impact of these modifications on sound production, advanced measurement techniques were required. One such technique is laser diffractive interferometry, which provides a powerful tool for investigating the fluid-dynamic and acoustic interactions within the instrument. In this study, a commercial Polytec Laser Doppler Vibrometer (LDV) with a Mach-Zehnder configuration was used. The LDV measures the Doppler frequency shift, which is proportional to changes in the laser optical path. Typically, LDV systems measure vibration velocities by detecting surface displacements. However, in this study, the Doppler signal results from the interaction between the laser and a turbulent flow, characterized by density fluctuations, which modifies the optical path, as the laser beam passes through it. Therefore, the measured signal is referred to as "pseudo-velocity". Changes in the optical path are due to variations in the refractive index, which change the wavelength of the laser as it travels through the turbulent region. The relationship between the refractive index and density fluctuations is crucial for interpreting the measured signal. Traditionally, the Gladstone-Dale relation is used to describe this relationship for fluid mixtures [2], but in this study, the more accurate Lorentz-Lorenz equation was considered [3, 4]. Temperature fluctuations can also affect the refractive index, however their contribution is considered negligible compared to pressure-induced variations, which dominate the measurement field [5]. This technique offers valuable insights into turbulent flow dynamics and acoustic wave interactions, both essential for understanding the behavior of complex systems such as musical instruments. By using such advanced methodologies, we can gain a deeper understanding of how modifications to the instrument, such as the introduction of periodic structures, influence the overall sound spectrum and performance. This provides a foundation for future innovations in instrument design and optimization.

The characterization of musical instruments, particularly woodwinds, has been extensively studied across various methodologies and technologies. These studies focus on the interactions between the player, the instrument, and the surrounding environment. A foundational reference in the study of woodwinds is provided by Fletcher et al., who explore the physical principles governing sound production, encompassing both theoretical and practical aspects [1]. Complementing this, Benade et al. examine the acoustical foundations of musical instruments, placing significant emphasis on the acoustic properties of woodwinds [6]. Wolfe et al. further investigate the interaction between the instrument and the player, modeling wind instruments as a combination of nonlinear elements (such as reeds or lips) alongside upstream and downstream ducts [7]. In addition, Lefebvre et al. present computational methods for analyzing woodwinds, including the Transmission-Matrix Method (TMM) for calculating input impedance and an optimization algorithm for determining tonehole positions and dimensions [8]. In addition, Lucchetta et al. apply Experimental Acoustic Modal Analysis (AMA) to a vintage tenor saxophone. This method measures Frequency Response Functions (FRFs) through acoustic pressure and volume velocity, identifying key modal parameters such as resonance frequencies and mode shapes. These insights enhance our understanding of the instrument's cavity modes and aid in validating analytical and numerical models [9]. Recent developments in laser-based techniques have provided new avenues for the non-invasive study of musical instruments. Bakarezos et al. explore the advantages of Electronic Speckle Pattern Interferometry (ESPI) for analyzing surface vibrations in instruments [10]. Huber et al. focus on scanning LDV for studying metal surface vibrations in guitars and organ reed pipes [11]. Beyond surface analysis, laser techniques are also used to study airflow and turbulence in woodwinds. Martarelli et al., Vanherzeele et al., and Castellini et al. highlight the use of LDV in detecting air fluctuations [12–15]. Mayrhofer et al., Woisetshlaeger et al. and Zipser et al. apply laser interferometry for frequency-resolved analysis of turbulent flows, further supported by uncertainty analysis from Castellini et al. [5, 16–19]. Additionally, Koeberl et al. and Kleine et al. demonstrate the effectiveness of interferometric techniques in visualizing compressible flows, contributing to a deeper understanding of air dynamics in woodwind instruments [20, 21]. These combined studies, ranging from traditional acoustic models to advanced optical measurements, provide a comprehensive framework for understanding and optimizing the acoustic performance of woodwind instruments.

## Material and Methods

### *Solid periodic ring structure*

The introduction of Solid Periodic Ring Structures (SPRS) inside a woodwind body represents a novel application. While the effects of periodic structures are well-understood in other domains, their impact on the emission properties of woodwind instruments has not yet been thoroughly investigated. To verify their effectiveness, we decided to use two simple configurations reported in Figure 1 which are compared to the unaltered ring-less SPRS reference. In the 2D case, the open lateral sides of each structure are sealed using a glass slide in order to avoid any pressure leak and, at the same time, allow the probe laser light to pass through the elements. The aim is to achieve an alteration of the instrument's timbre in a very high-frequency range, in such a way as to cut off the sharpest components and give the sound with a greater sense of roundness, enhancing its suitability for inclusion in classical performances. The distance  $d$  between periodic rings should only affect the highest harmonic content of the sound spectra.



**Fig. 1** Illustration of the two SPRS tested in the study. The left image (a) shows the axisymmetric (circular) configuration, while the right image (b) displays the 2D (square) configuration

In view of future optimizations, as a first attempt, we decided to vary the height  $t$  of the rings by 1 mm by leaving unaltered the other parameters, obtaining two configurations referred to as wide and narrow rings. Those configurations are then compared to a structure having flat walls (no rings) considered as a reference. Geometrical parameters are reported in Table 1. The above mentioned structure were designed in the Rhinoceros<sup>TM</sup> CAD environment and 3D printed using a 405nm LCD commercial printer.

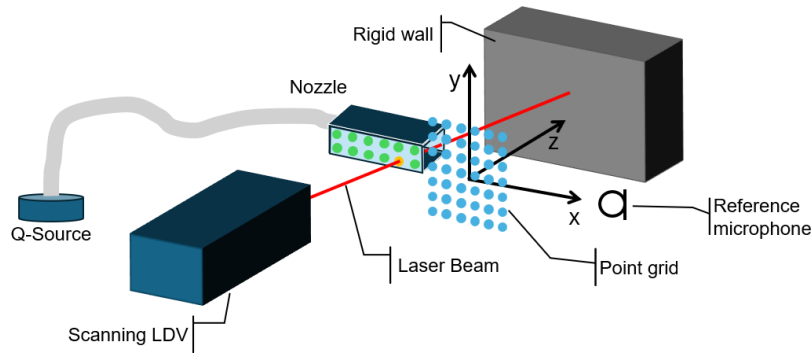
**Table 1** Dimensions of the three 3D printed structures under test.

|              | $t$ [mm] | $w$ [mm] | $d$ [mm] | $l$ [mm] | $\Phi$ [mm] |
|--------------|----------|----------|----------|----------|-------------|
| Flat walls   | 0        | 0        | 0        | 75       | 15          |
| Wide Rings   | 4        | 16       | 25       | 75       | 15          |
| Narrow Rings | 5        | 16       | 25       | 75       | 15          |

### *Experimental set-up*

The experimental analysis was conducted using a Polytec PSV-500 Scanning Laser Doppler Vibrometer (LDV) which is focused onto a fixed surface, a B&K microphone (model 378 B02 PCB, 1/2 inch) aligned to the nozzle axis to capture the acoustic field from the instrument under test and a monopole volume acceleration source (Simcenter Qsources Mid-Hig Frequency Volume Source, model Q-MHF) with an internal reference sensor connected to the SPRS as excitation. The experimental setup aimed to evaluate the acoustic pressure fluctuations and corresponding density variations within and around the SPRS introduced inside the instrument. The entire experimental setup, including the positioning of the LDV and microphone, is illustrated in Figure 2. The LDV system was configured to capture density fluctuations in the air caused by vibrating air column within the instrument. This is achieved by pointing the laser at a non-vibrating co-operative surface,

specifically a rigid wall covered by reflecting tape, with the sole function of reflecting as much light back as possible. Air density fluctuations directly affect the optical path of the laser beam, making it possible to map the dynamic behavior of the instrument's internal acoustics. The LDV captures pseudo-velocity by measuring the Doppler frequency shift, which corresponds to the changes in the optical path length due to the density variations in the vibrating air. The microphone was placed at a distance of 42 cm from the jet outlet in the near field, slightly downstream, to avoid interference with the airflow while maintaining a high Signal-to-Noise Ratio (SNR). A grid of measurement points was defined inside the SPRS section in the surrounding area to ensure a statistically valid dataset. Each point on this grid was scanned using the LDV to acquire pseudo-velocity data. The spectral bandwidth was set to 12.5 Hz – 10 kHz, with a spectral resolution of 12.5 Hz across 800 spectral lines. The sampling frequency was 25 kHz. A Hanning window was set for the acquisition to avoid leakage related issues. Moreover, each grid point measurement was averaged 16 times to enhance SNR. The excitation signal, which is a white noise, was provided using the built-in Polytec signal generator. The total number of measurement points varied depending on the specific sample under examination. For the cylindrical sample, 3479 points were measured, while for the prismatic sample, 3305 points were recorded. The wide rings and narrow rings samples were measured with 3262 and 3251 points respectively. Notably, only the grid corresponding to the cylindrical sample with flat walls (no rings) is rectangular, comprising 49 elements in the Y direction and 71 elements in the X direction. In contrast, the grids for the other samples have been custom-designed to meet specific experimental requirements. The primary goal for the experimental campaign was to compare the external acoustic pressure measured by the microphone with the internal pseudo-velocity data gathered by the LDV, allowing for a comprehensive evaluation of the interaction between the instrument's structural modifications and the corresponding acoustic field.



**Fig. 2** Overview of the experimental set-up for the microphone and vibrometry measurements

## Numerical Simulations

Numerical simulations are carried out by means of COMSOL Multiphysics Pressure Acoustic module, specific for modelling the propagation of sound waves through fluids. It operates in the frequency domain, allowing for the analysis of harmonic pressure variations in applications involving stationary fluids, like in the present case. Specifically, the sound pressure represents the deviation of the instantaneous pressure from the ambient atmospheric pressure, assuming a static fluid environment without significant flow.

Axisymmetric simplifying condition is assumed in the simulations: both geometry and boundary conditions approximately exhibit rotational invariance, and their analysis can be reduced to a 2D cross-sectional study.

Three types of boundary conditions are needed to characterize the device physics:

- Spherical radiation boundaries are imposed at the outer surface of the computational domain, assumed to be a sphere much larger than the device under test (DUT). In this way, a spherical-wave approximation is implied on the above boundaries: outgoing spherical sound waves can pass through the boundary without significant reflections. The only small discontinuity in the radiation boundary is given by the acoustic cylindrical waveguide carrying the acoustic power, but such perturbation occurs at large distance from the DUT, and, as a matter of fact, a negligible fraction of the injected acoustic power is radiated backward.
- Solid, reflective surfaces are assumed on the surface of the woodwind instrument: normal component of sound waves are prevented from penetrating the device as if encountering a rigid, unyielding wall.

- Wave-port condition is used to simulate the incident pressure field having the form of an acoustic wave injected into the feeding waveguide. The incident pressure field constitutes the input excitation of the problem and assumes the simple form of a plane-wave owing to its spatial uniformity in the section of the waveguide.

It is remarked that the wave-port is considered as transparent, meaning that waves possibly reflected back to it are perfectly absorbed. Such condition is satisfied, in case of plane-wave “radiation option”, if the reflected waves propagate near the normal to the port. Otherwise, rigorous multimode analysis is needed at the port, or at least, a proper perfectly-matched-layer (PML) setup. Such issue is not affecting significantly the present simulation, as, in fact, the waveguide section is smaller than typical acoustic wavelengths, and the transverse profile of the acoustic field is highly constrained.

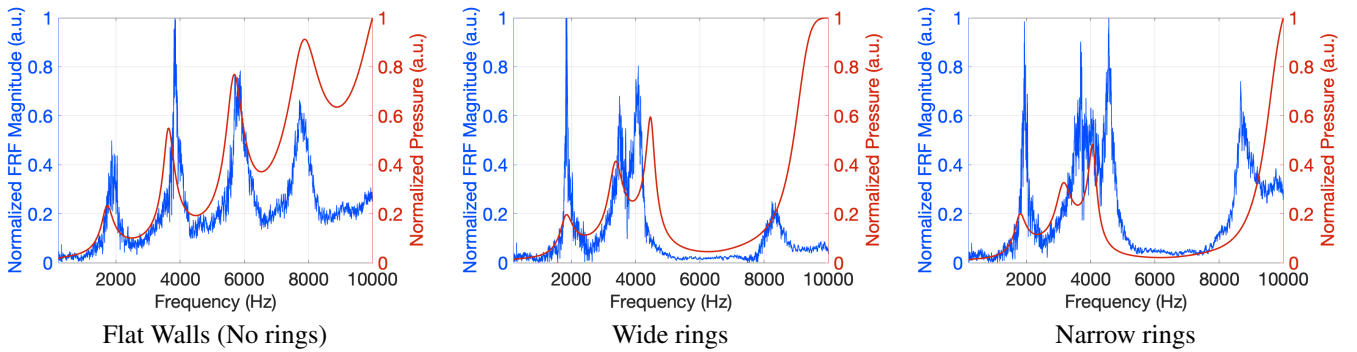
The acoustic pressure level is provided by the numerical solver throughout the computational domain. The acoustic pressure level, sampled at a point along the axis of the axisymmetric domain and located 40 cm from the woodwind instrument, is analyzed to illustrate the harmonic pressure variations within the system.

## Results

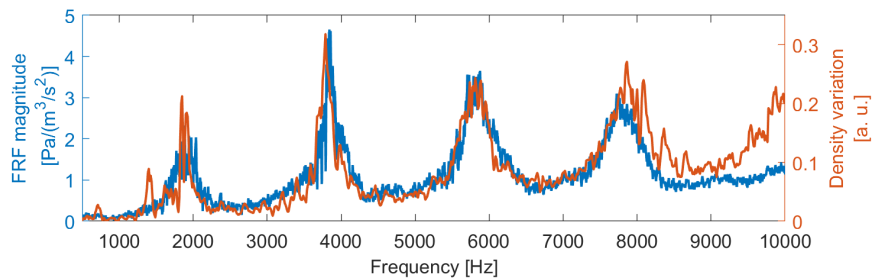
This section details the results of acoustic measurements conducted on various configurations of the SPRS. Each measurement type offers critical insights into how structural modifications influence the acoustic properties and behavior of the SPRS, thereby providing a thorough understanding of its performance under varying conditions.

Figure 3 illustrates the comparison of acoustic pressure levels across three configurations: flat walls (no rings), narrow rings, and wide rings. Each curve is normalized for clarity, allowing for a direct comparison of the acoustic responses. The results demonstrate how the presence and geometry of the rings significantly impact the acoustic response of the system.

- Flat walls (No Rings): The baseline configuration exhibits a broad frequency response, with relatively high levels of acoustic pressure across a wide range of frequencies. This indicates that the absence of structural modifications leads to less targeted wave interaction.



**Fig. 3** Comparison of acoustic pressure levels across three configurations: flat walls (no rings), wide rings and narrow rings (numerical data in red, experimental data in blue)



**Fig. 4** Comparison between the acoustic FRF measured by the microphone and the density variation measured by the LDV for the element with flat walls (no rings).

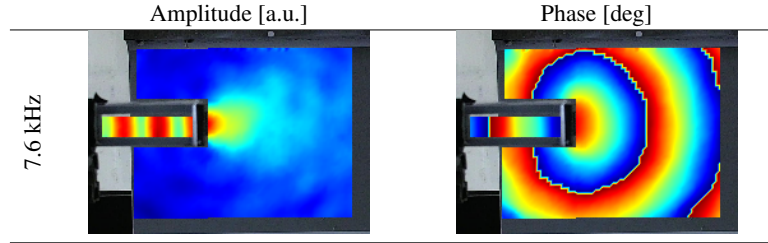


- **Narrow Rings:** The introduction of narrow rings in the geometry alters the frequency response, creating specific resonances that affect the amplitude of the acoustic pressure. These resonances appear as distinct peaks in the spectrum, showcasing how the geometry influences sound behavior.
- **Wide Rings:** The presence of wide rings produces a different frequency response compared to the narrow rings. The wider geometry shifts the frequency characteristics further, demonstrating a more pronounced impact on the high-frequency content of the acoustic signal.

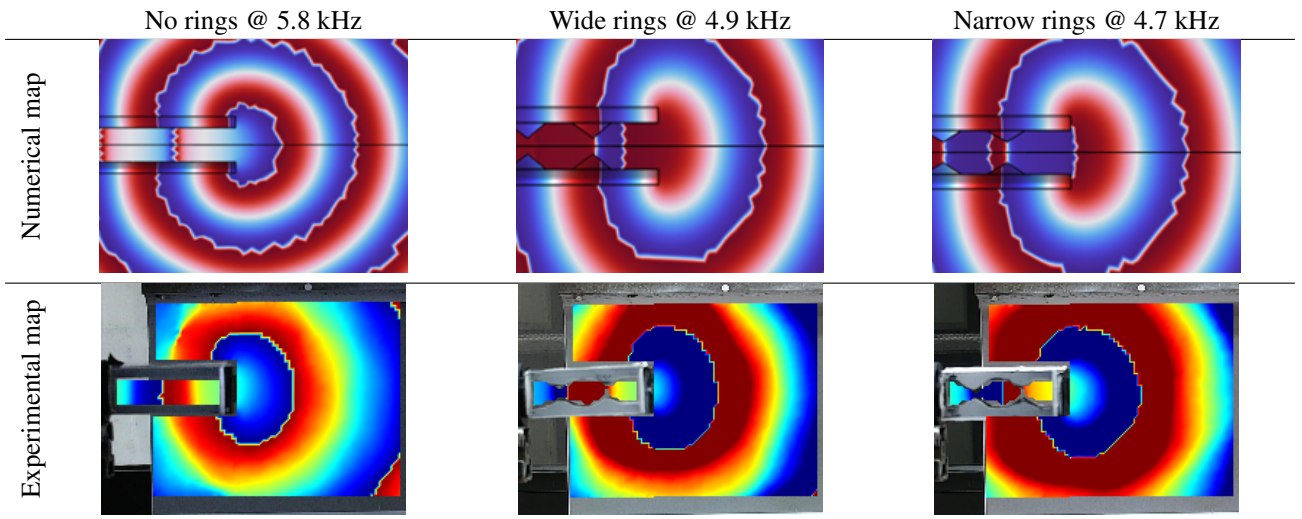
The comparison between microphone and LDV measurements is shown in Figure 4. Microphone measurements provides data on the external acoustic pressure, reflecting how sound propagates beyond the instrument. LDV measurements capture density variations associated with the acoustic wave. The resulting spectra reveal how these density changes correlate with the sound pressure levels measured by the microphone.

Figure 5 shows a representative example of the amplitude and phase map for the prismatic attenuator configuration (flat walls condition) at a frequency of 7.6 kHz. These maps serve as a reference point to evaluate the influence of the absence of rings on the acoustic behavior of the system.

Finally, in Figure 6, the numerical and experimental phase maps (color scale ranging from  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$ ) for the three configurations (no rings, wide rings and narrow rings) are compared at the fourth resonance frequency. Concerning the numerical model for the wide and narrow rings configurations, the phase maps reveal that nodal lines align with the constrictions, where a node emerges along the symmetry axis. This suggests that, at the frequency in question, sound waves concentrate around these areas due to the geometric changes, with waves being partially reflected and refracted at the constrictions. This behavior contrasts with the flat walls configuration (no rings), where the absence of physical obstructions causes the nodal lines to distribute differently. The experimental results follow a similar trend however exhibiting some discrepancies, which are primarily attributable to the 3D printing process. In fact, its accuracy is not sufficient to produce flawless constrictions in terms of geometry and roughness in the specimens, as in the numerical model. These irregularities affect the sound wave



**Fig. 5** Amplitude and phase maps for the prismatic attenuator configuration at the fourth resonance frequency (7.6 kHz).



**Fig. 6** Comparison between phase maps for wave propagation in the flat walls (no rings), wide rings and narrow rings structures calculated from the numerical model and experimentally measured.

propagation and the formation of nodal lines. Additionally, the acoustic pressure measurements are subjected to uncertainties caused by density fluctuations due to external factors, setup assembly and geometric imperfections of the specimen under test.

## Conclusion

This study investigated the effects of Solid Periodic Ring Structures (SPRS) introduction on the acoustic properties of woodwind instruments. Different geometrical configurations of periodic rings are explored evaluating their impact on the sound spectra and overall acoustic behavior. Both numerical predictions and experimental results highlighted the crucial influence of rings presence on the acoustic pressure levels and frequency response of the instruments. Moreover, their design significantly affects the acoustic response. The configurations with narrow and wide rings produced distinct resonances, leading to targeted modifications of the sound output. This results suggest that a careful design of ring geometry can enhance the quality of sound by attenuating undesirable harmonics, thereby aligning the instrument timbre with the preferences of classical musicians seeking a mellow tone. The correlation between external acoustic pressure and internal density variations, measured with a microphone and a LDV respectively, underscores the importance of understanding these interactions to optimize instrument design. Furthermore, the numerical simulations corroborated the experimental findings, illustrating the nodal patterns in sound wave propagation influenced by the SPRS. Although discrepancies were noted due to the limitations of the 3D printing process, the alignment of nodal lines with structural modifications affirms the potential of using numerical models to predict acoustic behaviors effectively. Future research should focus on refining the manufacturing process of the SPRS to enhance structural accuracy and improve acoustic consistency. Exploring additional configurations and materials for the periodic structures could further optimize sound characteristics.

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## Chapter 2

# Multi-Scale, Multi-Mode Damage Simulation for Miter Gates

Guofeng Qian, Haiwei Lin, Zhen Hu, and Michael D. Todd

**Abstract** Different failure modes can evolve simultaneously in large civil infrastructure systems like miter gates throughout their service life. However, most physics-based prognostic models focus solely on the evolution of a single damage mode, neglecting the significant interactions between different damage modes. This paper proposes a physics-based, multi-scale, multi-mode damage prognostic model applied to miter gate structures important to the inland waterways navigation corridor. The model simulates four common types of damage: two global damage modes (contact loss between the wall and gate and regional uniform corrosion), and two local damage modes (pitting corrosion and cracking). First, the two global damage modes are simulated using the finite element method by modifying the boundary conditions and element dimensions. Next, pitting corrosion is modeled at the local scale with a phase-field method, using the stress response from the global structural analysis model. A pit-to-crack transition is identified, incorporating information from both the corrosion simulation and the structural analysis models to indicate the transition process. Subsequently, the extended finite element method (XFEM) is applied to simulate localized crack growth. Finally, the growth of the local crack is used to update the global structural analysis through an iterative local-global model. The interactions between contact loss, regional uniform corrosion, pitting corrosion, and cracking are simulated in a holistic simulation framework, with the global and local models updated iteratively. This allows for the development of a systematic failure prognostic framework for large civil structural systems using physics-based analysis. The proposed method is demonstrated using a miter gate structural example to showcase its effectiveness.

**Keywords** Multi-Scale Simulation · Multi-Mode Damage Prognostics · Structural Health Monitoring · Local-Global Model

## Introduction

Multiple types of deterioration can occur simultaneously in civil structures throughout their decadal service lives [1, 2]. Taking the miter gate structure used in inland navigational lock systems as an example, multiple studies have been conducted on various deterioration types associated with the aging of these structures, including wall/boundary contact loss [3, 4], localized corrosion [5], and cracking [6, 7]. It is crucial to note that the evolution of different damage types influence each other. For instance, stress corrosion cracking can initiate from localized corrosion [8], and uniform corrosion can lead to stress redistribution within the gate, impacting gaps and crack formation [9]. Despite these interactions, the influence between different damage types has not been sufficiently studied to fully understand their interdependencies [10]. This study investigates a multi-scale, multi-mode damage simulation framework to capture the inter-damage influence on damage evolution. In the following sections, we introduce the simulation framework, detailing each component, including macro structural analysis, meso-scale corrosion evolution, and local-global crack propagation.

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Guofeng Qian · Haiwei Lin · Michael D. Todd

Department of Structural Engineering, University of California San Diego, La Jolla, CA 92037

e-mail: [g4qian@ucsd.edu](mailto:g4qian@ucsd.edu); [hal145@ucsd.edu](mailto:hal145@ucsd.edu); [mdtodd@ucsd.edu](mailto:mdtodd@ucsd.edu)

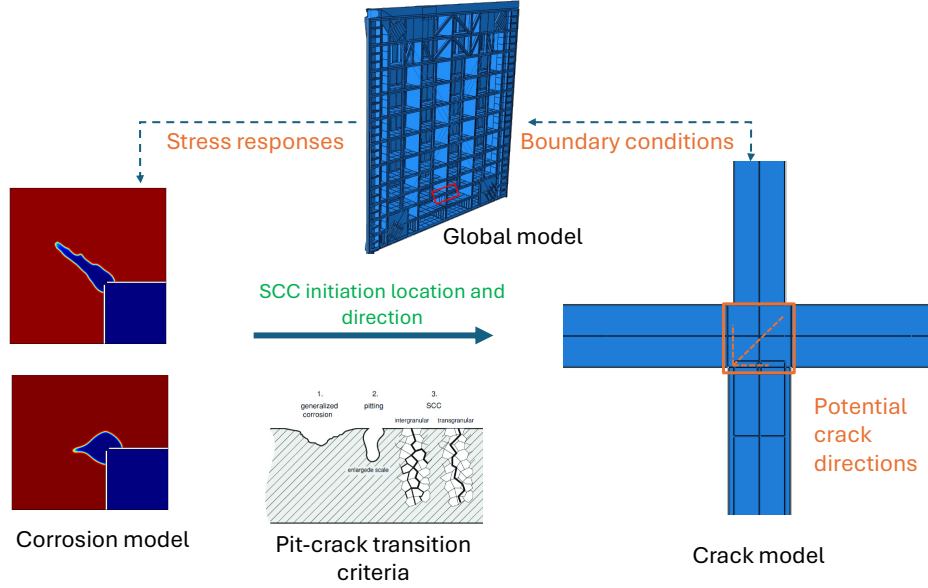
Zhen Hu

Department of Industrial and Manufacturing Systems Engineering, University of Michigan-Dearborn, Dearborn, MI 48128

e-mail: [zhennhu@umich.edu](mailto:zhennhu@umich.edu)

## Multi-Scale Multi-Mode Damage Simulation Framework

Fig. 1 illustrates the overall framework. Due to the length scale differences between the structure and the localized crack and corrosion damage, maintaining the same level of fidelity in the simulation is computationally prohibitive. To address this challenge, we implemented a multi-scale approach.



**Fig. 1** Overall framework for multi-mode damage simulation

The framework consists of a macroscale structural analysis finite element model that serves as the global model. This global model simulates the strain response of the structure under operational loads and accounts for global damage, including structural thickness loss due to uniform corrosion and contact loss between the quoin block and the wall.

On one hand, the global model provides the local strain response, which acts as the mechanical load input to the meso-scale corrosion evolution model. On the other hand, the global model interacts with the local crack model by iteratively adjusting the boundary conditions until convergence is achieved. The corrosion model also informs the crack model by identifying the stress corrosion cracking (SCC) initiation location and direction.

### Macro-Scale Structural Analysis

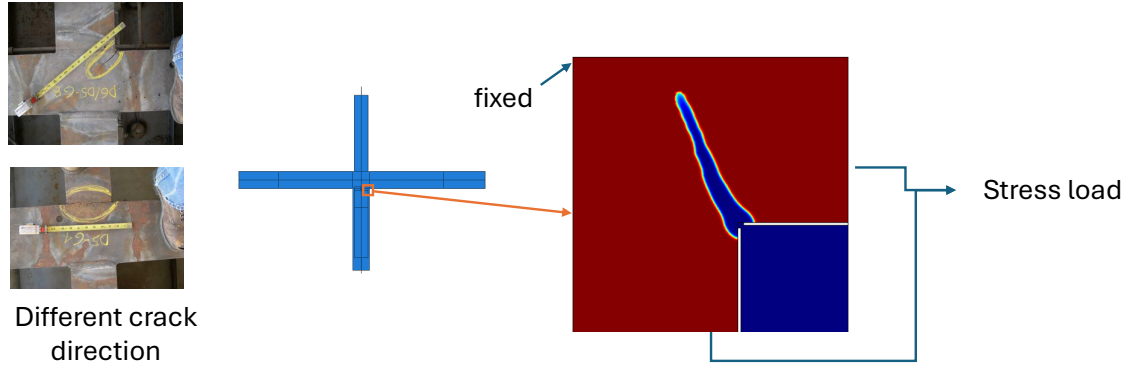
To provide the correct strain response to the local model, it is essential to simulate both gap damage and uniform corrosion in the global model. In the finite element (FE) model, the gap is represented by a free boundary condition, simulating the loss of contact between the gate and the wall quoin block near the bottom of the gate [4].

For uniform corrosion, we simulate its impact on the structure by reducing the thickness of the structural components in the FE model. Given that different areas of the structure experience varying degrees of corrosion, we divided the structure into three distinct zones: the upper zone exposed to air, the middle zone subject to fluctuating water level exposure, and the lower zone always submerged in water. The corresponding mass losses in these zones are denoted as  $t_1$ ,  $t_2$ , and  $t_3$ , where  $t_2 > t_3 > t_1$ . By simulating these varying thickness losses, we can effectively analyze effects of uniform corrosion on different regions of the structure.

### Meso-Scale Corrosion Evolution and Crack Initiation

As most SCC initiates from the corner of the intersection of webs, as shown on the left side of Fig. 2, we further perform a phase-field corrosion simulation for the corner area. This model is capable of predicting SCC initiation [5] as well as

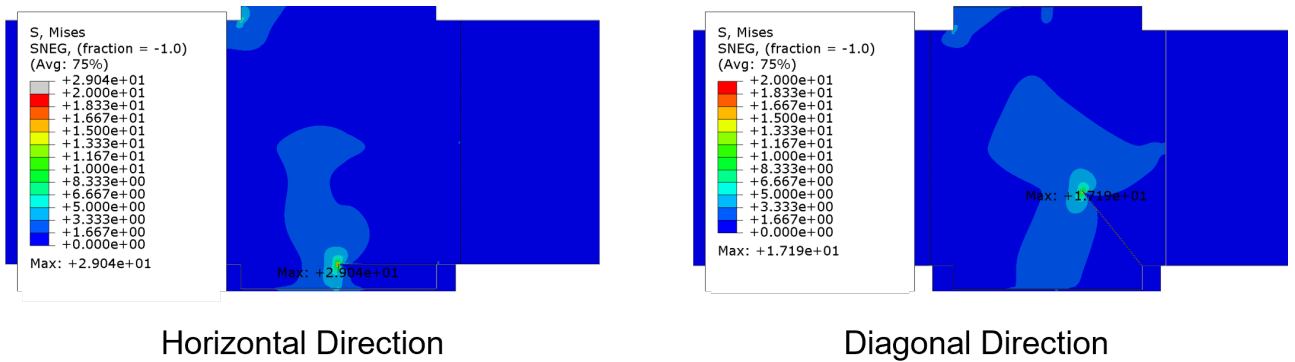
determining the initial direction of SCC propagation when subjected to different mechanical loads from the top surface and the right surface. The simulation utilizes a 2D model since the flange is a thin plate and is therefore represented using two-dimensional shell elements. Once SCC initiates at a specific time and propagates in a particular direction, the crack progression will be further analyzed using the crack model described in the next section.



**Fig. 2** Localized corrosion simulation and crack initiation direction.

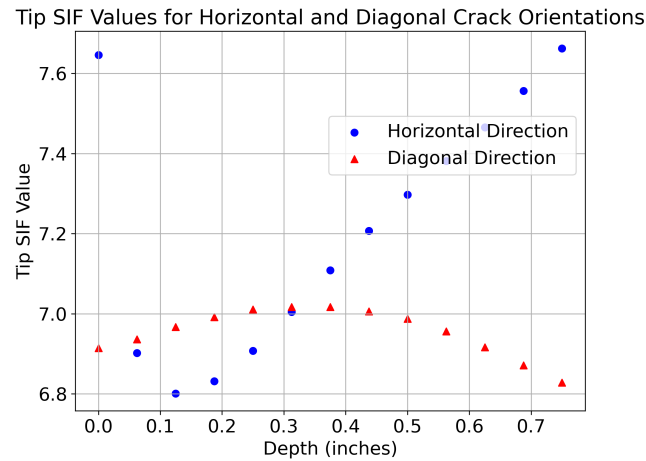
### SCC and stress intensity factor Simulation

Based on the different directions induced by corrosion, we conducted simulations of crack propagation under various conditions and calculated the stress intensity factor (SIF) at the crack tip by iteratively coupling the global model and the local model, exchanging boundary conditions between them [7]. For illustration, we present two SCC directions initiating from the same location with the same crack length.



**Fig. 3** Stress Distribution around Tip from Two Different Initiation Directions

Figure 3 displays the von Mises stress distribution maps for SCC in horizontal and vertical directions. We observe that the horizontal direction exhibits higher stress values as the depth increases, while the diagonal direction shows lower stress values under these conditions. Additionally, Fig. 4 presents the maximum SIF values along the depth near the crack tip. Consistent with the stress results, SCC in the horizontal direction has higher SIF values, whereas SCC in the diagonal direction has lower SIF values. These findings indicate that SCC in different directions results in varying propagation velocities, even when the initiation location and crack length are identical.



**Fig. 4** Tip SIF Values for Different Cracks

## Conclusions

This paper proposes a systematic framework to simulate four interdependent damage models in the miter gate structure. The studied damage modes include contact loss, thickness loss due to uniform corrosion, localized corrosion, and cracking. The interactions between these different damage types were effectively captured using a multi-scale simulation. The developed framework will be used for failure prognostics of miter gates considering various uncertainty sources.

**Acknowledgments** This work is under the support of the US Army Corps of Engineers through the U.S. Army Engineer Research and Development Center Research Cooperative Agreement W9132T-22-20014. The support is gratefully acknowledged.

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## Chapter 3

# Likelihood-Free Bayesian Inference for Localized Corrosion in Miter Gates

Guofeng Qian, Jice Zeng, Zhen Hu, and Michael D. Todd

**Abstract** Multi-scale corrosion simulation models have the capability of predicting the evolution of corrosion damage on large civil structures like miter gates under different environmental and operational conditions. However, updating such models using Bayesian methods is challenging because they often involve heterogeneous uncertainty sources and/or complex model architectures, making the likelihood function analytically intractable. This paper proposes a likelihood-free probabilistic model updating method for a complex multi-scale corrosion model based on a conditional invertible neural network (cINN). The multi-scale physics-based model consists of a local phase-field corrosion model, a global structural analysis model, and a statistical distribution model. Due to the lack of observational data, the cINN is first trained with simulation data. Once trained, the model can update multiple key parameters using image observations. Specifically, pit dimensions are identified from these observations using a convolutional neural network (CNN). Features extracted from the CNN are then used to infer the posterior distribution with the pre-trained cINN, eliminating the need to evaluate the likelihood function. The inference results show improved accuracy and significantly lower computational costs compared to the existing likelihood-free model updating methods. The proposed method is demonstrated on a miter gate example to showcase its effectiveness.

**Keywords** Likelihood-Free Bayesian Inference · Conditional Invertible Neural Network · Structural Health Monitoring

## Introduction

Localized corrosion is one of the most dangerous forms of deterioration in large metal civil structures, not only because it can initiate stress corrosion cracking (SCC) [1], but also due to its difficulty to be detected. Vision-based damage detection and diagnostics [2, 3] have been extensively studied in the structural health monitoring (SHM) literature. This approach has the advantage of efficiently detecting local damage at a low cost, making it well-suited for localized corrosion detection.

Traditional Bayesian inference requires the evaluation of an explicit likelihood function to extract information from available observations. However, in physics-based simulations, such likelihood functions are often intractable due to the complexity of coupled partial differential equations [4]. In these cases, likelihood-free or simulation-based Bayesian inference approaches are necessary. However, these methods typically require large numbers of simulations, making the inference process computationally expensive. Flow-based amortized Bayesian inference [5, 6] offers a more efficient alternative, providing fast inference compared to other simulation-based approaches. The goal of this paper is to perform amortized Bayesian model updating for complex multi-scale corrosion simulation. In what follows, we first introduce the forward simulation model of the localized corrosion damage. Then, we further introduce the inference process with the conditional invertible neural network. The inference results are compared with a classical approximate Bayesian computation method followed by the conclusion.

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Guofeng Qian · Michael D. Todd

Department of Structural Engineering, University of California San Diego, La Jolla, CA 92037

e-mail: [g4qian@ucsd.edu](mailto:g4qian@ucsd.edu); [mdtodd@ucsd.edu](mailto:mdtodd@ucsd.edu)

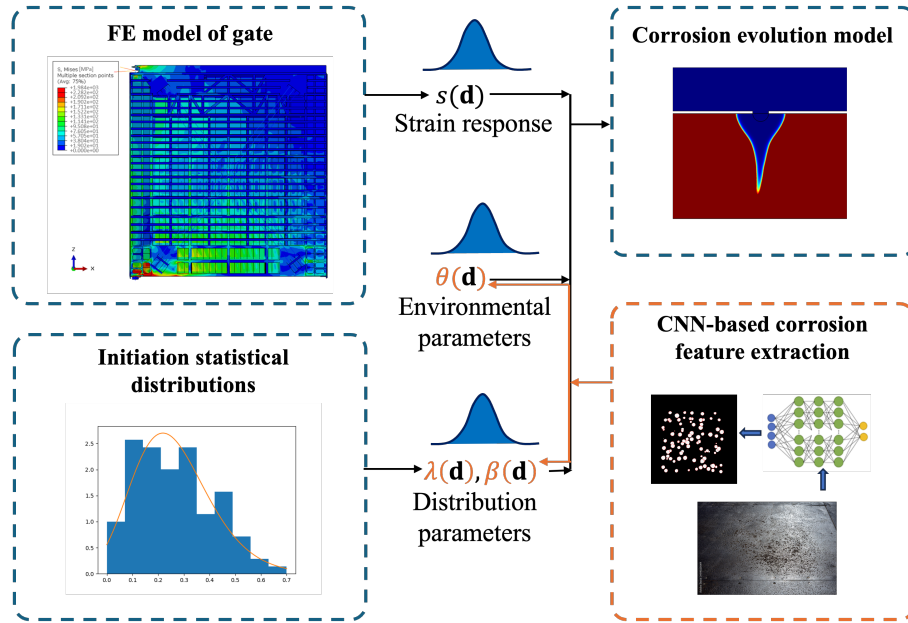
Zhen Hu

Department of Industrial and Manufacturing Systems Engineering, University of Michigan-Dearborn, Dearborn, MI 48128

e-mail: [patrickjice@gmail.com](mailto:patrickjice@gmail.com); [zhennhu@umich.edu](mailto:zhennhu@umich.edu)

## Forward Model for Multiscale Corrosion Simulation

The overall framework of this study is shown in Figure 1. There are three parts in the forward multi-scale corrosion simulation model. The first part is the macroscale finite element (FE) model of the Greenup miter gate structure (as shown in the upper left of Fig. 1). This model was previously validated with strain gauge measurements [7]. This high-fidelity FE model consists of about 65,000 elements. Most of these elements are 3D shell elements since the most common component in the gate is the thin steel plate. The diagonals and anchor bars are simulated with beam-column elements. The 4-node, reduced integration linear quadrilateral elements are primarily used with 3-node full-integration linear triangular elements for complex geometry. This model is responsible for simulating the strain response  $s(\mathbf{d})$  of the gate at any spatial location  $\mathbf{d}$  for any given operating load, hydrostatic pressure, and gap length due to the loss of contact between the quoin block and the wall [8]. The local strain response  $s$  provides the strain load to the local corrosion evolution model, as illustrated in the upper right of Fig. 1. This localized corrosion evolution model is constructed using the phase-field method with four partial differential equations together, i.e., Butler-Volmer equation for kinetics, diffusion equation, Poisson's equation, and mechanical equilibrium [9]. The model is able to simulation the corrosion under mechanical strain  $s(\mathbf{d})$  and environmental parameter  $\theta(\mathbf{d})$  which describes the influence from the local environment at  $\mathbf{d}$  on the corrosion reaction.



**Fig. 1** Overall framework for multi-scale corrosion simulation

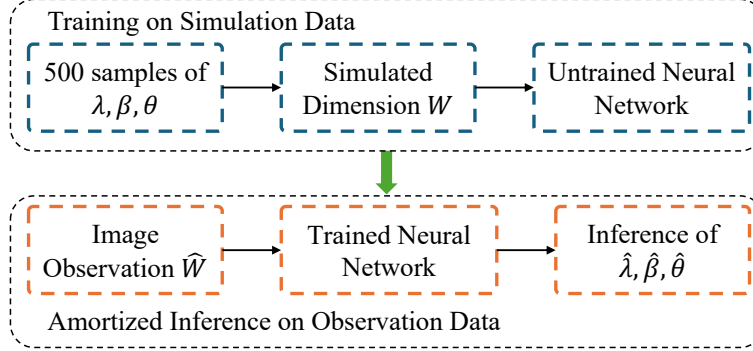
Corrosion initiation must be simulated to inform the pit evolution process. The initiation process is influenced by different factors including temperature, chemical composition, material, and potential [10] at local position  $\mathbf{d}$ . However, detailed mechanism of corrosion initiation is still under active research [11], making it difficult to simulate it using physics-based simulation models. Therefore, we simulate the initiation process statistically as shown in the lower left in Fig. 1. A Weibull distribution is employed for the modeling of the corrosion initiation time. This distribution is governed by two parameters, namely  $\lambda(\mathbf{d}), \beta(\mathbf{d})$ . Combined with the environmental parameter  $\theta(\mathbf{d})$ , we simulate localized corrosion growth using a high-fidelity corrosion model within a multiscale approach applied to the large structure.

## Likelihood-Free Bayesian Inference with Invertible Neural Network

There are three parts in the forward model as mentioned above. The strain response  $s$  is measurable with strain gauges. Even it is not practical to measure every location on the gate, the gate model can be validated and updated with a few strain measurements at the global model level. However, some of the parameter values are difficult to measure. The environmental parameter  $\theta$  is almost impossible to measure locally as it is influenced by the temperature, water composition, material, and

other factors. The distribution parameters  $\lambda, \beta$  are not measurable directly either. Therefore, we implemented likelihood-free approach to infer these immeasurable parameters, shown in orange color in Fig. 1, including  $\lambda, \beta$  and  $\theta$ .

We implemented amortized Bayesian inference with conditional invertible neural network (cINN) based on normalizing flow. Due to its invertibility property, we can leverage the relationship from model parameters to observable results from simulations. By inverting this relation, we can perform the inference from observable results back to model parameters in an amortized way [5]. The flow of amortized inference is shown in Fig. 2.



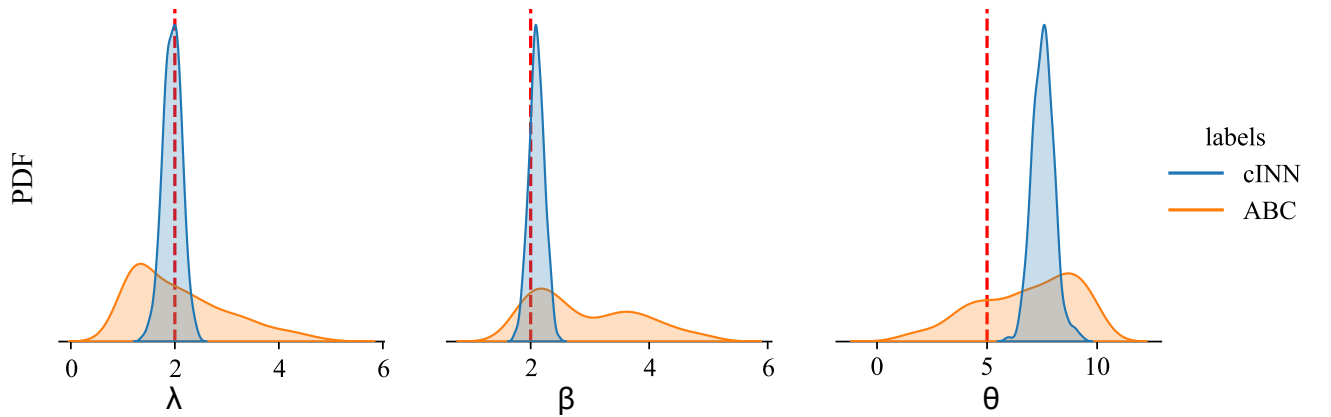
**Fig. 2** Amortized inference approach

The pit opening width distribution,  $W$ , is selected as the observation from synthetic images since actual gate data were not available. A CNN-based feature extraction technique has been developed [12].

As shown in Figure 2 above, the proposed cINN is trained on the parameters  $\lambda, \beta, \theta$  and the observation  $W$ . After training with five hundred samples of input parameters and simulation results, cINN is able to perform the amortized Bayesian inference for  $\hat{\lambda}, \hat{\beta}, \hat{\theta}$  given one real observation  $\hat{W}$ .

## Comparison of inference results

The classic approximate Bayesian computation (ABC) is also performed as a reference with the same number of simulations used to train cINN. The simulation results from prior samples are compared with the observation to determine whether the sample is accepted or rejected. The inference time for each observation will take about 4 hours. While the cINN learns the relationship between different parameters and according to results. Although the training stage takes 5 hours, the inference can be completed within seconds for one observation. This allows fast inference for large amounts of observation which is not possible with traditional likelihood-free inference methods like ABC.



**Fig. 3** Posterior comparison between cINN and ABC

The comparison of inference results is shown in Fig. 3. It is obvious that the cINN approach has more accurate and confident posterior for  $\lambda$  and  $\beta$  than ABC approach. However, for environmental parameter  $\theta$ , both approaches has some



bias for larger  $\theta$  in the inference results. This bias probably comes from the observation bias as small pits might not be detected. As a result of detecting pits later, simulation-based approaches will infer a faster growth speed to match the big pit width [13]. In conclusion, cINN-based inference method has more accurate inference results with less variance. It reduces the inference time significantly compared to other approaches. It might be more sensitive to observation bias as well.

## Conclusions

The application of amortized Bayesian inference with cINN for model updating for physics-based multiscale corrosion simulations offers several notable advantages, which are summarized as below

- **Efficiency:** The model can be inferred and updated within seconds, making it highly suitable for time-sensitive applications.
- **Accuracy:** cINN consistently outperforms other likelihood-free methods, such as ABC, by significantly reducing the variance in inference.
- **Potential Limitation:** While cINN provides reliable results, it may sometimes exhibit overconfidence in its inferences, particularly in cases where observation bias is present.

Overall, cINN proves to be an effective tool for complex corrosion modeling that supports an SHM process for miter gates. With further refinement, pit observation bias will be considered more carefully.

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## Chapter 4

# A Methodology to Manage the Complexity of a Nonlinear Multi-body Digital Twin in Railway Applications



Betty Auzanneau, Emeline Sadoulet-Reboul, Emmanuel Foltête, Guillaume Ham-Livet, and Scott Cogan

**Abstract** In the railway industry, digital twins based on nonlinear multibody simulations are developed to provide decision-making support for bogie design. A recurring challenge in the use of numerical simulations is to find an acceptable compromise between the complexity of the models for the different components – in terms of their capacity to adequately reflect specific physical effects – and the associated computational burden, so as to ensure a sufficiently accurate representation of the global dynamic behavior a train. This requires an investigation into the extent to which, for instance, the stiffness and damping characteristics of a component depend on parameters such as the displacement amplitude, the frequency or the load direction. Intensive experimental studies are one strategy for increasing the knowledge of components, but they are costly to implement or incomplete when parametric dependencies are not explored. It is then necessary to determine which factors need to be controlled for each component, and to what extent, in order to prioritize them. In this context, the objective of the proposed work is to investigate different modelling strategies integrating the complexity of a physical model in different ways, and to discriminate between the effectiveness of these strategies in faithfully reproducing the dynamic responses of a structure. Two approaches are investigated in particular to study a real yaw damper component of a motor bogie. Firstly, a rheological model representing the dynamic behavior of the damper is proposed, and the parameters of this model are identified on the basis of characterization tests. This physics-based model is integrated into dynamic simulations for sensitivity studies in order to identify the influential elements that need to be controlled to validate the dynamic model. A simulation-based sensitivity indicator is developed as a tool to implement the necessary complexity for each component and to optimize the design of the required experimental studies and the methodology is illustrated for the yaw damper component mounted on a locomotive's motor bogie. In a second phase, an alternative model based on a multi-layer perceptron neural network is proposed to improve the computational efficiency of the digital twin.

**Keywords** Sensitivity indicator · Nonlinear multibody simulations · Parametric model · Data-driven model · Railway application

## Introduction

The railway industry is a sector in which the search for performance and safety is an essential part of the design process. The growing use of digital twins is opening up new and unexplored avenues both in the preliminary design phase and in the final phase when railway vehicles are homologated. The bogies at the interface between the bodies and the track are critical components ensuring the stability and comfort of trains, and nonlinear multibody simulations provide assistance in the choice of suspension components at the design stage in order to guarantee good overall dynamic behavior of the train. A recurring challenge in the use of numerical simulations is to find an acceptable compromise between the complexity of the models and the associated computational burden. Simplified models may not accurately capture complex dynamic behavior, while models that are too detailed may incur prohibitive computational costs and be unusable in practice. In this context, the purpose of the research work is to investigate the level of model complexity required to validate the dynamic model of a

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Betty Auzanneau · Emeline Sadoulet-Reboul · Emmanuel Foltête · Scott Cogan  
Department of Applied Mechanics Institut FEMTO-ST, Université de Franche-Comté, SUPMICROTECH, CNRS, 25000 Besançon, France  
e-mail: [betty.auzanneau@femto](mailto:betty.auzanneau@femto); [emeline.sadoulet-reboul@univ-fcomte.fr](mailto:emeline.sadoulet-reboul@univ-fcomte.fr); [emmanuel.foltete@univ-fcomte.fr](mailto:emmanuel.foltete@univ-fcomte.fr); [scott.cogan@univ-fcomte.fr](mailto:scott.cogan@univ-fcomte.fr)

Betty Auzanneau · Guillaume Ham-Livet  
ALSTOM Transport SA71200 Le Creusot, France  
e-mail: [betty.auzanneau@femto](mailto:betty.auzanneau@femto); [guillaume.ham@alstomgroup.com](mailto:guillaume.ham@alstomgroup.com)

railway vehicle and a specific component of a bogie, in this case a yaw damper, is considered. This question is both related to the problem of modeling, largely investigated in the railway domain by [1] and of quantifying model-form error that has been a topic of recent investigations, for example [2]. Two approaches are considered in this study. First, a physics-based model is developed using a rheological model for the yaw damper. Even if some references introduce 3D geometrical models for such components [3] [4] [5], models based on combination of viscous elements, springs, and dashpots are widely used, and well-adapted for multibody simulations [6] [7] [8]. An automated procedure is used to determine a set of feasible values for model parameters in order to approximate the measured behavior of the component and a decision-support indicator for the model complexity is introduced. Secondly, an AI-based method is introduced to develop a simplified model, with reduced computational times. Neural networks are an emerging strategy to define input-output relations for complex components presenting nonlinear behaviors [9] [10]. The behavior of the damper is represented by a multi layer perceptron neural network expressing the complex relationship between force, displacement, velocity and acceleration at the component interfaces and experimental validation is performed in order to evaluate the model's fidelity-to-data.

## Presentation of the Study-case

The yaw damper presented in Figure 1 is characterized using a traction-compression machine. A series of 30 tests are performed imposing a mono-harmonic displacement excitation at various amplitudes and frequencies and the measured force data show a nonlinear behavior of the component depending of the amplitude and frequency of the excitation. The post-processing of the experimental results allows to compute the global mechanical characteristics (the macro values): the complex dynamic stiffness amplitude and the loss factor. To differentiate the component behavior with the macro values, a third characteristic based on the force-displacement hysteresis shape is added.



**Fig. 1** Experimental bench used for the yaw damper characterization (a). Rheological model of the yaw damper (b).

## A Complexity Level Decision Support Indicator For The Rheological Model

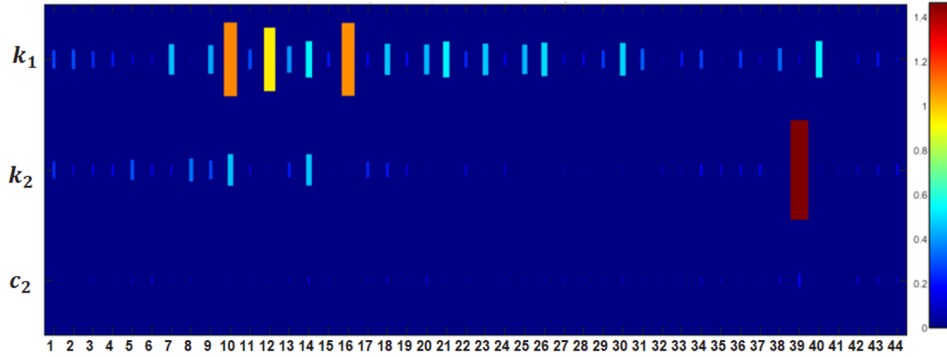
### Determination of the rheological model parameters

Figure (1b) illustrates the yaw damper modeling using a rheological model inspired by two conventional Maxwell models. The nonlinear damping parameter  $F_{c1}$  is expressed using the experimental values. A set of values for the linear stiffness parameters  $k_1$  and  $k_2$ , as well as the linear damping parameter  $c_2$ , representative of the damper behavior, can be found using an exploratory phase that is defined in three steps. The process begins with the generation step is the generation of a sample of the  $k_1$   $k_2$  and  $c_2$  values. Next, the simulated behavior of the component is calculated using these values and the resulting macro values are obtained. Finally, the latest values are compared with accepted tolerances and the sample is rejected if declared unsuitable.

### Modeling level decision support indicator

The yaw damper model is integrated in an early design phase of a high-speed train locomotive model and 44 output features of interest are defined to quantify the operational safety and the comfort of the railway vehicle. The objective is to determine the influence of the rheological model parameters on the output features and to determine whether the model can be simplified. A variance-based sensitivity indicator using the previous sample based macro values tolerances is computed and allows to rank the parameter influence on the outputs. The results are displayed as a color matrix in Figure 2 and demonstrates that,

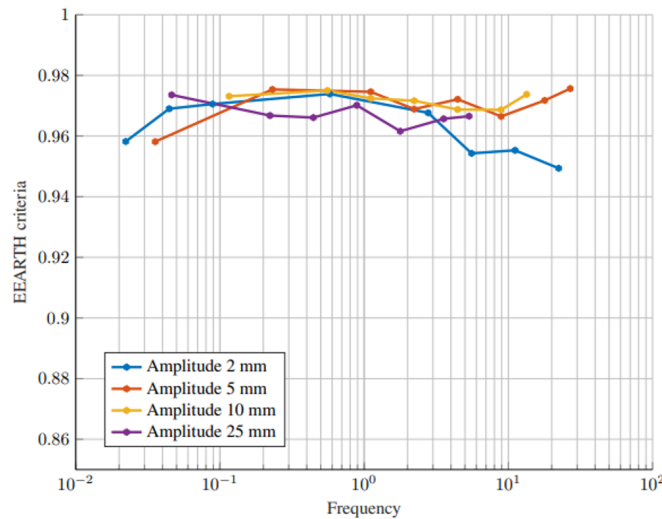
for outputs 10, 12, and 16, the parameter  $k_1$  has the largest impact. This suggests that the model could be simplified by eliminating the secondary Maxwell branch; which is not actually possible due to the output 39 indicator which indicates that the parameter  $k_2$  has the highest influence.



**Fig. 2** Variance-based sensitivity indicator values visualized as a color matrix with parameters on the rows and outputs on the columns.

## A Neural Network Modeling

An alternative AI-based modeling approach is highlighted. The force is expressed as a function of the displacement  $x$ , the velocity  $v$  and the acceleration  $a$ :  $F = g(x, v, a)$  and  $g$  is determined using an artificial neural network. The retained architecture is a multi layer perceptron composed of 3 inputs in the input layers: the displacement  $x$ , velocity  $v$  and acceleration  $a$ , 6 hidden layers, each with 30 neurons, and 1 output in the output layer: the force  $F$ . The Rectified Linear Unit (ReLU) function is selected as the activation function. Since the 30 data sets have a different number of values, their contents are interpolated to get 6 000 points in each one leading to 180 000 input values for the neural network. The whole set of input values is divided in the training data and the validation data in the proportion 80-20 and the Adam (adaptive moment estimator) optimizer is chosen. The resulting neural network model is validated using the original experimental data sets and the EEARTH score (Enhanced Error Assessment of Response Time Histories) metric, introduced in [11], is used to quantify the distance between the simulated and experimental forces as illustrated on Figure 3. The scores, above 0.949, demonstrate that the neural network model accurately captures the behavior of the component on the measured domain.



**Fig. 3** Validation of the neural network with the computation of the EEARTH criteria for the 30 sets as a function of the excitation frequency.

## Conclusion

A method for the search of the level of complexity required to validate a model in railway dynamics has been introduced. It relies on the determination of a decision-support indicator based on sensitivity analyses and it is applied to a yaw damper rheological model mounted on a motor bogie. A model developed using a multilayer perceptron neural network has been introduced that fully represents the component behavior within its training domain. The use of this type of black-box modeling disregarding any physical characteristics and time efficient is a good alternative when it comes to finding an acceptable compromise between the model complexity and the computational load.

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## Chapter 5

# Degradation Model Correction of Miter Gates Through Synthesis of Hybrid Modeling with Recursive Bayesian State Estimation

Zhao Zhao, Yichao Zeng, Michael D. Todd, and Zhen Hu

**Abstract** Modeling the degradation behavior of miter gates is of importance for risk-informed maintenance decision-making. Due to the complicated nature of the damage mechanisms and the lack of understanding them clearly, degradation models are usually developed based on simplifications and assumptions. As a result, these models may not accurately represent the true degradation behavior. To improve the accuracy of empirical degradation models of miter gates and ensure reliable failure prognostics, this paper proposes a novel degradation model correction framework by integrating hybrid modeling techniques with recursive Bayesian state estimation. The proposed work consists of two phases: the offline training phase and the online updating phase. During the offline training phase, the unknown model bias term is modeled as a random noise with uncertain standard deviation. Synthetic data are then generated using a physical model based on prior knowledge of the uncertain model parameters and model bias. The synthetic data is utilized to train a conditional Invertible Neural Network (an inference network). In the online updating phase, the trained inference network is used to obtain posterior samples. Meanwhile, the Gaussian mixture model and dual particle filter techniques are utilized to recursively update posterior samples, thereby improving the estimation accuracy. Subsequently, model bias is inversely estimated using the degradation model, which uses the updated model parameters and the gap length from the inference model. Finally, a regression model is utilized to correct the biases inherent in the simplified degradation model. Results of a case study show that the proposed framework can effectively improve the accuracy of the degradation model for failure prognostics of miter gates.

**Keywords** Degradation model · Miter gates · Model Uncertainty · Parameter Uncertainty

## Introduction

Miter gates are key components in inland waterway lock systems to enable the smooth transit of barges and ships [1–3]. Despite their critical role, a significant number of these gates have surpassed their intended design lifespan of 50 years, leading to an urgent need for effective repair and maintenance strategies. Thus, it is imperative to understand the deterioration mechanism of miter gates [4, 5]. Although a simplified degradation model for miter gates has been developed to describe the gap growth based on abstracted inspection data in recent years [6], its prediction error is inevitable because of many inherent assumptions and simplifications.

To improve the accuracy and reliability of the degradation model and further better predict failure conditions, this study presents a hybrid modeling technique with recursive Bayesian state estimation, comprehensively accounting for both parameter and model uncertainties.

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Zhao Zhao · Yichao Zeng · Zhen Hu

Department of Industrial and Manufacturing Systems Engineering, University of Michigan-Dearborn, Dearborn, MI 48128, USA

e-mail: [zhaozhao@umich.edu](mailto:zhaozhao@umich.edu); [zyichao@umich.edu](mailto:zyichao@umich.edu); [zhennhu@umich.edu](mailto:zhennhu@umich.edu)

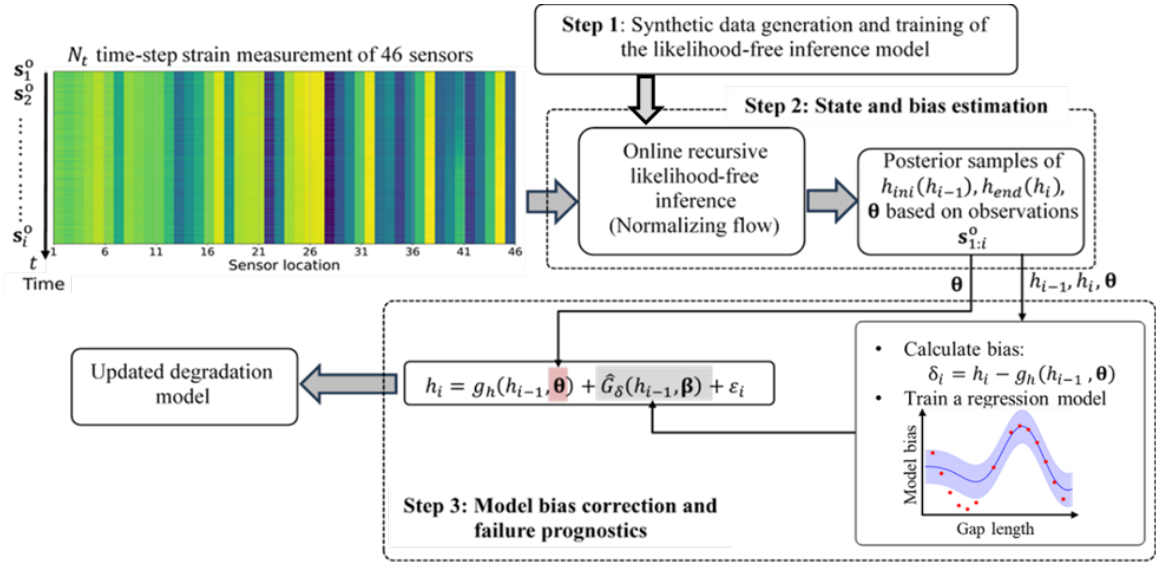
Michael D. Todd

Department of Structural Eng., University of California, San Diego, 9500 Gilman Drive, La Jolla CA 92093, USA

e-mail: [mdtodd@ucsd.edu](mailto:mdtodd@ucsd.edu)

## Proposed Method

The whole technical route of the proposed method is shown in Fig. 1, which consists of three main steps [7]. In the first step, we generate synthetic data based on prior knowledge of model parameters and bias through a simplified degradation model. We assume that model bias is affected by gap length, which is incorporated into the formula for measuring unobservable gap lengths. Additionally, the prior is constrained to preserve the model's physical meaning. By applying the two-level degradation model, we can generate synthetic strain data for specified model parameters, gap lengths, and biases. Thus, the training dataset is structured with gap length, model bias, and model parameters as inputs, and strain data as outputs. After that, we focus on the offline training for a likelihood-free inference model. The training datasets are first processed through a specially designed summary network. Then, an inference network, such as a conditional Invertible Neural Network (cINN), is used to perform joint training for inference. Following that, in Step 2, posterior samples of gap length and model parameters are generated by plugging new observations into the trained summary network and cINN. To further refine these posterior samples, two techniques, including a Gaussian Mixture Model (GMM) and Dual Particle Filtering, are employed for recursive sample updates over a long-term monitoring period. Based on the recursive updating of the posterior distributions of gap length and model parameters, a numerical procedure is developed to estimate the bias of the degradation model. In Step 3, two regression models, including a Gaussian process regression (GPR) model and a polynomial regression (PR) model, are employed to capture the relationship between gap length and model bias. Finally, the predictive accuracy of degradation curves is improved by utilizing the updated model parameters and the model bias identified through the GPR model or the PR model. More details of the method can be found in Ref. [7].



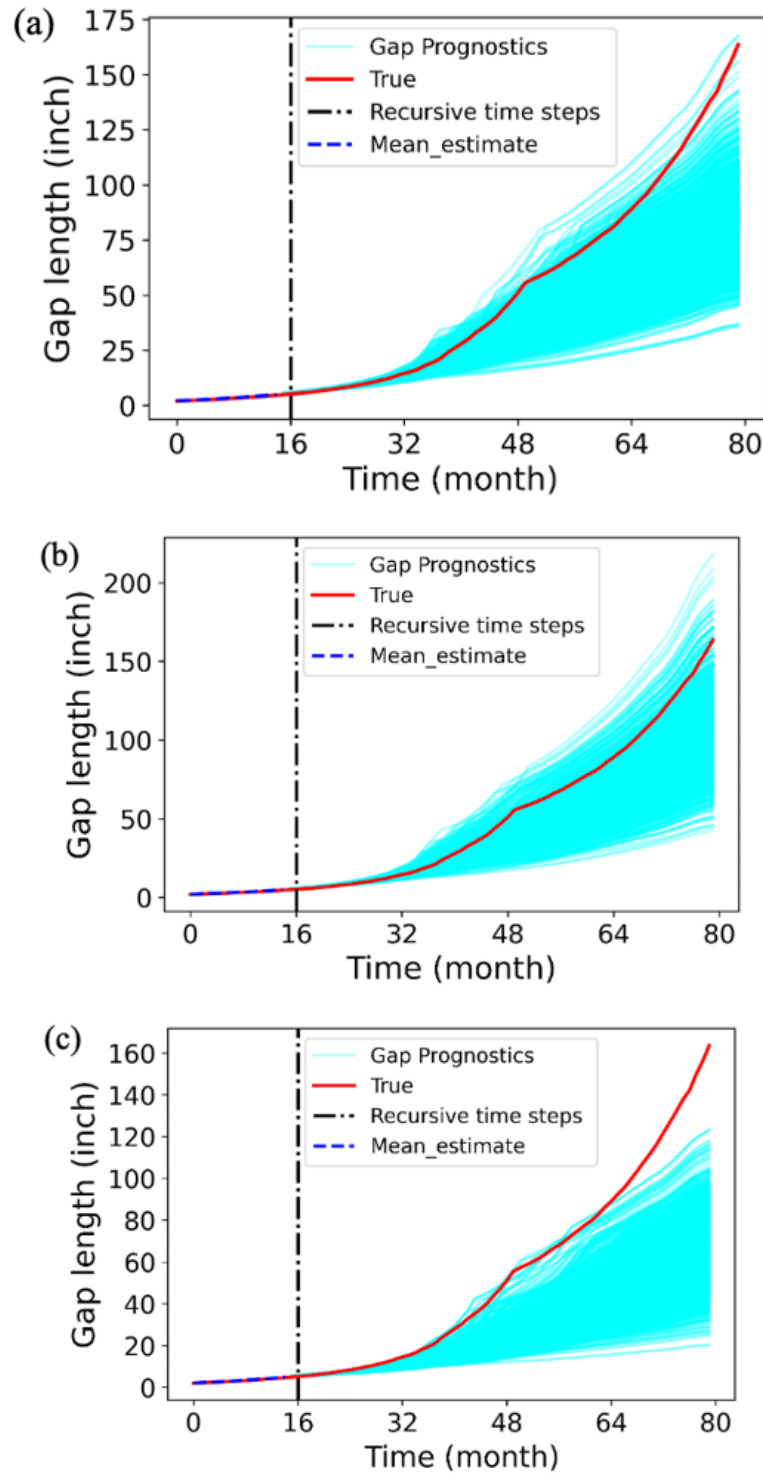
**Fig. 1** Schematic diagram of the proposed methodology [7].

## Application To A Miter Gate

To assess the effectiveness of the proposed model, synthetic strain measurements are initially generated using model parameters from a true degradation model. While the true degradation model is considered unknown, our proposed method aims to infer the true parameters and model bias for given strain data. We then evaluate the accuracy of failure prognostics, such as predicting the remaining useful life (RUL), by comparing the performance of the updated degradation model against that of the original simplified degradation model.

Fig. 2 presents a comparison of gap length predictions, which accounts for model bias using respectively GPR model and PR model, against predictions made without considering model bias [7]. The prognostic outcomes for gap length obtained from both the GPR and PR models exhibit a high degree of accuracy and closely follow the true degradation curve, outperforming the original model that ignores model bias.

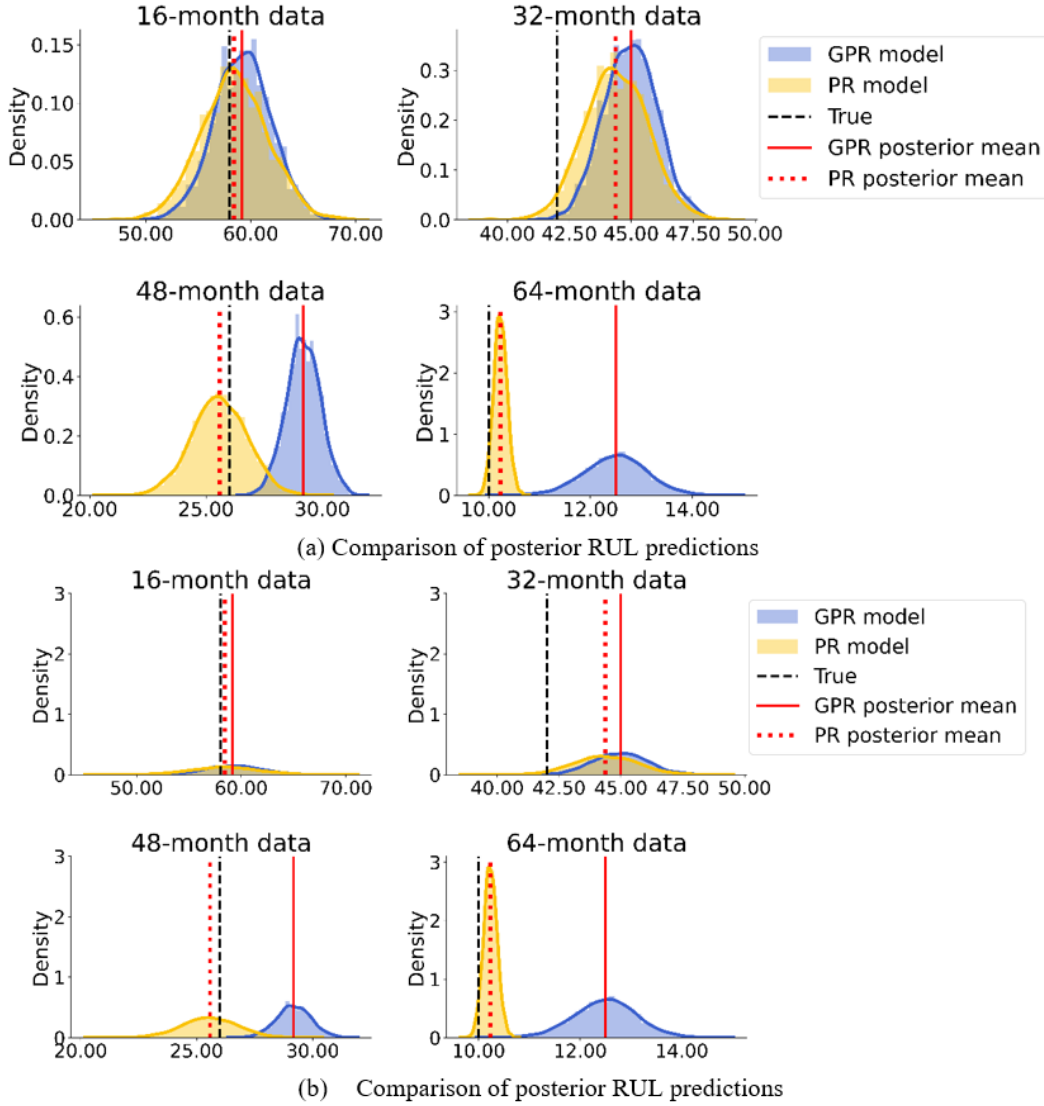




**Fig. 2** Prediction of degradation curves by 16-month data: (a) GPR model prediction; (b) PR model prediction; (c) prediction without model bias [7].

Fig. 3 presents the density distributions of RUL estimates using both GPR and PR models across four different data durations (16-month, 32-month, 48-month, and 64-month). Across all scenarios, the estimates of the PR model consistently align more closely with the true RUL compared to that of the GPR model. This consistent performance highlights the PR model's overall superior accuracy and reliability in RUL estimation compared to the GPR model.





**Fig. 3** Estimated posterior distribution of RUL [7].

## Conclusion

This study proposes an innovative method for degradation models correction through a hybrid modeling with recursive Bayesian state estimation. The primary objective is to simultaneously account for model parameter uncertainty and model bias and improve the accuracy of model predictions and failure prognostics, specifically for miter gate degradation models. The proposed method has two phases: an offline training phase and an online recursive updating phase. During the offline training phase, synthetic data is generated to jointly train a summary network and a cINN model. In the online recursive updating phase, the gap length, bias parameter, and degradation model parameters are estimated by the trained cINN based on measured strain data. The model updating process is implemented recursively to refine the posterior samples of the model parameters. This refinement is achieved through the application of dual particle filtering in conjunction with a GMM to enhance the accuracy of the estimations. The model bias is inversely estimated by the gap length estimated from the cINN model and recursively updated model parameters. A GPR model and a PR model are used to establish the relation between gap length and model bias. The failure prognostics (e.g., prediction of degradation curves and RUL estimate) is improved by the updated model parameters and model bias captured by the GPR and PR models. The proposed method is applied to a miter gate structure and the results show that the proposed approach significantly outperforms a simplified degradation model that ignores model bias, which highlights the importance of accounting for model bias in degradation models.

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## Chapter 6

# Uncertainty Quantification of Fatigue Damage from Random Vibration and Sine-sweep

Arup Maji

**Abstract** A damage-based approach is often used for different types of fatigue failure to quantify the relative severity of vibration specifications. Efficient numerical codes based on structural response allow rapid evaluation of the effect of both input and model uncertainties on ‘damage’. Damage index is computed using a fatigue damage model (S-N curve) based on stress cycles. Because these approaches are approximate, and are the only means to provide engineered guidance, quantification of uncertainties is essential to their use.

Comparing the severity of various vibration test specifications is complicated since the Auto-spectral Density (ASD) can be higher or lower at different frequencies. Test durations are also different. An added complexity is that sine-sweep test specifications (from directly coupled engine vibration) can be prescribed to be concurrent with random vibration ASDs (structure borne vibration). A system or component may be subject to vibration during transport and handling vs. during launch or flight. It then becomes important to quantify the consequences of various test specifications to prioritize testing, or to ensure tests done to previous specifications can be used to ascertain safety when requirements and usage has changed later in the development phase.

The paper presents the technical basis for comparing fatigue damage from various types of vibration test specifications. Codes developed and associated theoretical approximations will be discussed, along with associated uncertainty quantifications.

**Keywords** Fatigue damage · Vibration · Specification · Modal

## Background

Amplification at resonance is  $Q = 1/(2\xi)$ , where  $\xi$  is the viscous damping factor. The half-power (3db) bandwidth ( $\Delta f$ ) captures the frequency band around the resonant frequency  $f_n$  within which most of the energy associated with resonance is concentrated:  $\Delta f = 2\xi f_n = f_n/Q$ . The basic premise of equivalencing sine-tone into random-vibe and vice-versa is to look at each frequency, consider the energy within the half power (3db) bandwidth ( $\Delta f$ ), and evaluate the equivalent imparted damage. Fackler [1] provides detailed discussion on the technical aspects relevant to such equivalencing and shows that the peak amplitude  $G_s$  of the sine-tone is proportional to  $\sqrt{(\pi Q G_{xx} f_n/2)}$ , where  $G_{xx}$  is the Auto-spectral Density (ASD) of the equivalent random-vibe. Details of the equivalencing process accounting for the different duration for which the sine-sweep dwells in each octave band have been provided by Maji [2]. It uses  $R_f$  to account for the Raleigh distribution of peaks in a narrow-band random to get

$$G_s = R_f \sqrt{(\pi G_{xx} f_n / (2Q))} \quad (1)$$

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*This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.*

Arup Maji  
Sandia National Laboratories, Albuquerque, NM  
e-mail: [amaji@sandia.gov](mailto:amaji@sandia.gov)

In the research reported here, a sequence of numerical simulations with progressively greater complexity were undertaken to quantify the uncertainty associated with each approximation:

- Using a Single Narrow-band Random Input.
- Influence of Adjacent Bands on Conservatism.
- Convergence of damage with increasing duration.
- Effect of sampling frequency for pure sinusoid.
- Full-Frequency range, one bandwidth at a time ( $R_f = 1.0$ , unconservative).
- Full-Frequency range one bandwidth at a time ( $R_f = 1.4$ , conservative).
- Full-Frequency range with one bandwidth at a time (most accurate, band-specific  $R_f$ ).
- All bandwidths simultaneously, with realistic simulated transfer functions.

## Numerical Simulations

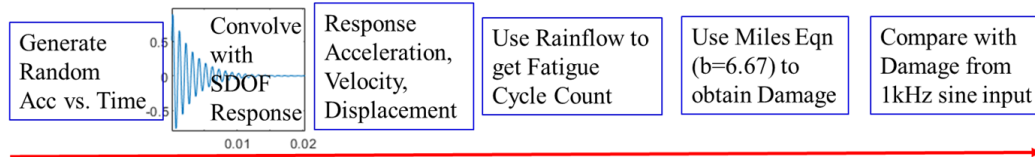
Table 1 shows the various simulations with increasing complexity that are reported in this document.

**Table 1** Numerical Simulations with Increasing Complexity

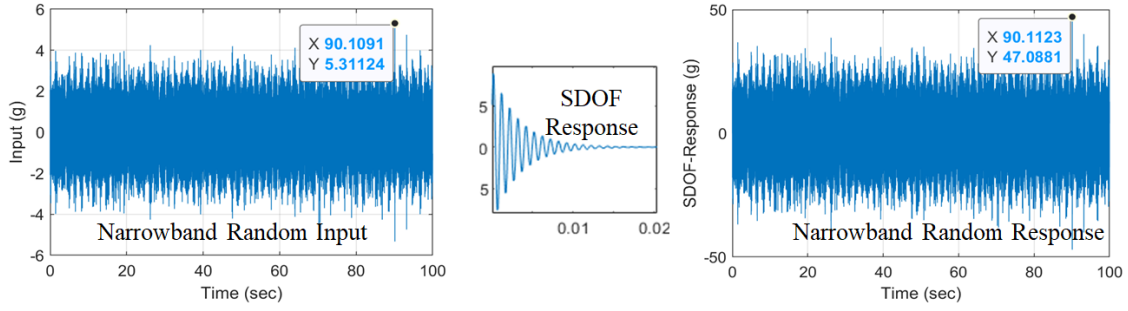
| Simulation Type                                  | Simulation # | Input Frequency Bands | Response Modes      | Analysis Type    | Objective of Simulation                             |
|--------------------------------------------------|--------------|-----------------------|---------------------|------------------|-----------------------------------------------------|
| Numerical Simulations with Increasing Complexity | Simulation 1 | 950-1050Hz            | 1000Hz              | single band      | Narrow-band accuracy                                |
|                                                  | Simulation 2 | 950-1050Hz, & broader | 1000Hz & 1100Hz     | single band      | Effect of adjacent bands and off-center resonance   |
|                                                  | Simulation 3 | 950-1050Hz            | 1000Hz              | single band      | Statistical Variation & Convergence                 |
|                                                  | Simulation 4 | 1000-4000Hz           | center of each band | 1 band at a time | Accuracy with $R_f = 1.0$                           |
|                                                  | Simulation 5 | 1000-4000Hz           | center of each band | 1 band at a time | Accuracy with $R_f = 1.4$                           |
|                                                  | Simulation 6 | 1000-4000Hz           | center of each band | 1 band at a time | Accuracy with band-specific $R_f$                   |
|                                                  | Simulation 7 | 1000-4000Hz           | Transfer Function   | concurrent       | Realistic input, response and full range simulation |

### Simulation #1: Using a Single Narrow-band Random Input

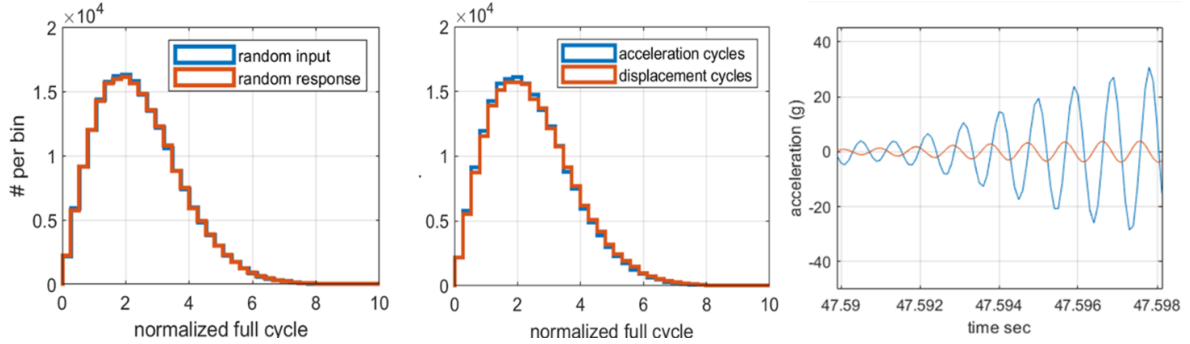
The objective of this simulation was to examine and validate that the response due to narrow-band input (acceleration, relative acceleration, and displacement) all have a Rayleigh distribution, which forms the fundamental basis for the factor  $R_f$ . In this validation process, a narrow-band random-vibration (950–1050 Hz with ASD = 0.01  $\text{g}^2/\text{Hz}$ ) was convolved with the transfer function of a 1 kHz single degree of freedom (SDOF) system. Figure 1 shows the steps by which the damage due to a narrow-band ASD can be determined for a 1 kHz SDOF response. Damage ( $\propto \Delta\sigma^b$ ) is computed based on Miner's rule and the number of fatigue cycles obtained using MATLAB™ 'rainflow'. Figure 2 shows the random-vibe input convolved with the transfer function at 1 kHz and the resulting response (approximate amplification of 10 using damping  $\xi = 0.05$ ). Displacement was obtained by numerical integration of the acceleration followed by high-pass filtering to eliminate the low-frequency ( $< 10$  Hz) drift. Figure 3 shows that the input and response accelerations and response displacement all have Rayleigh distributions which relates to the parameter  $R_f$  in the conversion process [2]. The same is true for relative



**Fig. 1** Schematic of simulation for a single narrow-band input.



**Fig. 2** Narrow-band random input and response via convolution.



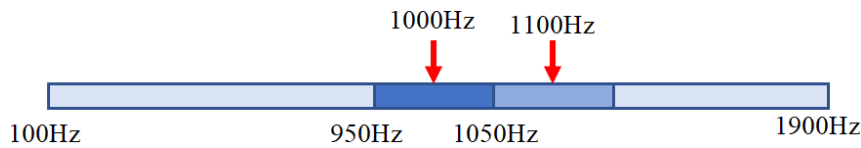
**Fig. 3** Normalized distribution of peaks in narrow-band input and response.

acceleration (response minus input) since the input and response acceleration have a phase offset of  $\pi$  (See Figure 3, right), so the peak acceleration remains the same as that of the response acceleration.

#### Simulation #2 Influence of Adjacent Bands on Conservatism

The objectives of this simulation were to validate the equivalencing equation for a single narrow-band and to investigate the effect of inputs from adjacent bands. The process of equivalencing is based on converting the sine-tone in each octave-spaced band (3dB bandwidth) into an equivalent ASD. This is achieved by setting equal the damage due to the response of a SDOF structure due to these two inputs (sine-tone centered in that band and the equivalent random-vibe ASD). Dividing the overall frequency range of interest into octave-bands is necessary to account for the fact that the underlying random-vibration ASD and the sinusoid are varying with frequency and time of test.

It is also necessary to recognize that a SDOF structure also responds to inputs outside of the narrow-band. It will be shown in the results below that a SDOF that has a natural frequency at the edge of a frequency bin will likely show greater conservatism from the equivalent ASD. These two issues were therefore explored with four sets of simulations. Figure 4 helps understand the adjacent bands and the SDOF resonances.



**Fig. 4** Adjacent bands and SDOF resonant frequencies considered in simulation (arrows).

A 1kHz sinusoid of peak amplitude 1.772g and a narrow-band ASD ( $0.01\text{g}^2/\text{Hz}$ ) between 950Hz and 1050Hz were compared first. A sampling rate of 10000Hz was used and the length of simulation was 100 seconds. These two inputs are equivalent per [Eq.1] and should therefore lead to the same damage to a SDOF with natural frequency  $f_n = 1000\text{Hz}$ . This is shown to be true in rows 1 and 2 in Table 2; the damages from the two inputs are essentially the same (difference by 0.03dB only).

Next, the natural frequency of the SDOF was changed to 1100Hz (outside of the narrow-band and centered approximately on the next adjacent narrow-band). Note that since the SDOF is outside of the input band-width, response is very low, and

the consequent damages are lower by -5.2dB and -6.9dB respectively (rows 3 and 4 in Table 2). It is worth noting that the SDOF at 1100Hz responds more to the sine-equivalent ASD than the sine-tone input at 1kHz and the consequent damage is 3.7 times higher ( $1.7e^{15}$  vs.  $4.6e^{14}$ ). Even though these values are much lower than the damage due to the SDOF centered at 1kHz ( $9.3e^{16}$ ), due to the nonlinear nature of damage (exponent  $b = 6.67$ ), small differences in response amplitude result in significant differences in damage. As a simple example, if one is to think of the responses of the 1000Hz and 1100Hz SDOF as  $a$  and  $b$ , where  $a \gg b$ , then the damage due to both of them together  $\propto (a+b)^{6.67}$  can be equal to  $(1+0.1)^{6.67} = 1.89$  or  $(1+0.05)^{6.67} = 1.38$ , or 0.5dB lower even though  $b \ll a$ .

Therefore, damage due to concurrent inputs at 1000Hz and 1100Hz to a SDOF at 950Hz were examined and the result shown in rows 5 and 6 of Table 2 show that the resulting total damage is now higher (than row 1 for the SDOF at 1000Hz only) for both the sine-equivalent ASD and the sinusoid input (2.52dB and 1.82dB, respectively). More importantly, damage from the sine-equivalent ASD is 0.7dB higher than that from the sinusoid.

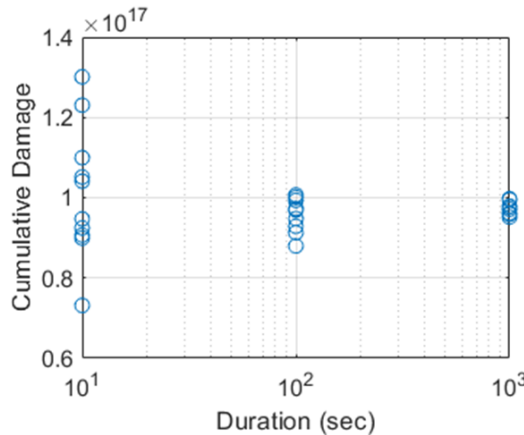
The last simulation extended the bandwidth of the ASD over a wider band (100-1900Hz), while keeping the ASD magnitude the same ( $0.01g^2/Hz$ ). Note that this results in the damage increasing by  $10\times$  to  $9.3e^{17}$ . This is expected according to theory, since the response of a SDOF to a wide-band ASD is  $\sqrt{2}\times$  higher than that due to a narrow-band ASD, consequently the damage is 3dB higher. In real tests, inputs will exist in all of the various narrow-bands, and these will affect the response at each resonant frequency of the structure of interest. The resulting increase in damage is disproportionately higher for the equivalent random-vibe input vs. the sinusoid input as per the results in the previous paragraphs.

**Table 2** Comparison of Damage from Wide vs. Narrow Band and Off-centered SDOF Frequency.

| Type of Input                  | Frequency (Hz) | ASD $g^2/Hz$ | Sinetone-Peak (g) | Inputs (kHz) | Damage     | Difference (dB) |
|--------------------------------|----------------|--------------|-------------------|--------------|------------|-----------------|
| Narrow-Band ASD, centered SDOF | 950-1050       | 0.01         |                   | 1.0          | $9.30E+16$ | 0               |
| Sinusoid only, centered SDOF   | $1.00E+03$     |              | 1.77              | 1.0          | $9.50E+16$ | 0.03            |
| Narrow-Band ASD, centered SDOF | 950-1050       | 0.01         |                   | 1.1          | $1.70E+15$ | -5.2            |
| Sinusoid only, centered SDOF   | $1.00E+03$     |              | 1.77              | 1.1          | $4.60E+14$ | -6.9            |
| Narrow-Band ASD, centered SDOF | 950-1050       | 0.01         |                   | 1.0&1.1      | $6.42E+17$ | 2.52            |
| Sinusoid only, 2 SDOFs         | $1.00E+03$     |              | 1.77              | 1.0&1.1      | $3.76E+17$ | 1.82            |
| Wide-Band ASD, centered SDOF   | 100-1900       | 0.01         |                   | 1.1          | $9.30E+17$ | 3.0             |

### Simulation #3 Convergence of Damage with Increasing Duration

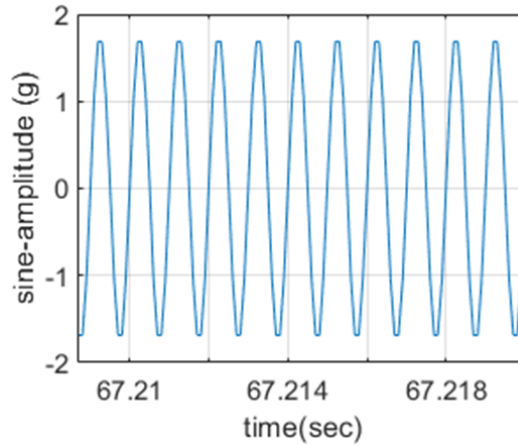
In order to examine the convergence of ‘damage’ from simulations of different duration, the previous simulation was repeated 10 times each for durations of 10sec, 100sec and 1000sec to examine the statistical variation. The damage values from the 10 and 1000 second duration were multiplied and divided by 10 respectively to make them comparable to those of the 100sec simulations. Figure 5 shows the progressive convergence with increased length of simulation. Comparing the highest and lowest value for each duration in Figure 5 shows that the variation for 10sec, 100sec and 1000sec simulations were 0.75dB, 0.18dB and 0.06dB, respectively. This result can provide guidance on the length of simulation necessary to achieve a certain



**Fig. 5** Convergence of damage with increasing duration.

level of accuracy; simulation longer than 100sec would not be worth the added time and associated cost if an accuracy of 0.18dB is acceptable.

A 100sec simulation of the equivalent sine-tone (peak = 1.772g) provided a damage of  $9.4 \times 10^{16}$ . It is important to note that even at a sampling rate of 10000Hz ( $10 \times f_n$ ) the peak of the sine-tone is clipped (Figure 6); peak is 1.69 vs. actual value of 1.77. The clipping would vary depending on the phase of the sine-tone and cause a small decrease in damage  $\propto (\text{amplitude})^b$ .



**Fig. 6** Effect of sampling frequency for pure sinusoid.

#### Simulation #4 Full-Frequency Range One Bandwidth at a Time ( $R_f = 1.0$ , conservative)

The next set of simulations involved using inputs over the entire frequency range of interest (1-4kHz), but for only one bandwidth at a time with the SDOF's natural frequency  $f_n$  at the center of the respective bands. Table 3 shows the input sine-sweep and the random-vibe ASD test specifications. The Sine-tone specification processed one octave band at a time to examine the accuracy of the process for each octave band. Note that in this simulation, the duration of dwell in each band and actual modal response of the structure were not considered, it simply examined the validity of [Eq.1] in providing a conservative estimate of equivalent ASD for the sine-tone in each band before a full-up simulation (simultaneous with all frequency bands) would be done.

**Table 3** Inputs for Broadband Simulation.

| Freq (Hz) | ASD ( $\text{g}^2/\text{Hz}$ ) | Freq (Hz) | Sinusoid Peak (g) |
|-----------|--------------------------------|-----------|-------------------|
| 1000      | 0.0025                         | 1000      | 1.5               |
| 4000      | 0.0018                         | 1600      | 2                 |
|           |                                | 2100      | 1                 |
|           |                                | 2500      | 1                 |
|           |                                | 3200      | 1.9               |
|           |                                | 4000      | 2.2               |

Table 4 shows the octave bands and columns along with the octave-band input sine-tone and the random-vibe ASD test specifications. For each octave band the equivalent ASD for the sine-tone in that octave band was calculated using [Eq.1]. The conservative limit value of  $R_f = 1.0$  was first used followed by more accurate values of  $R_f$  in subsequent simulations. A damage coefficient  $b = 6.67$  was used throughout since it is more commonly used and more sensitive than  $b = 4.0$ . The simulation involved the following steps:

Acceleration vs. time data was generated for each octave band using a sampling rate of  $10 \times$  the band's mid-frequency. For each band  $10^6$  instances of time were used. So for the sixth band with a mid-frequency of 1710Hz, the duration of simulation was  $10^6 / (10 \times 1710) = 58.5$  seconds. This was done for each of the sine-tone peak, original ASDs, equivalent ASDs (shown in the 3<sup>rd</sup>, 4<sup>th</sup> and 5<sup>th</sup> columns in Table 4, respectively) and for the sum of ASDs (sum of the 4<sup>th</sup> and 5<sup>th</sup> columns).

These acceleration vs. time data were combined as follows:

- Signal A: Acceleration response synthesized from the sine tone was added to the acceleration response synthesized from the original ASD (total\_sine\_acc).



- Signal B: Acceleration response synthesized from the sine-tone equivalent ASD was added to the acceleration response synthesized from the original ASD (total\_acc).
- Signal C: Acceleration response synthesized from the sum of the sine-tone equivalent ASD and the original ASD (totalASD\_acc).

Each of these three accelerations vs. time data were convolved with the SDOF impulse response centered at each octave band frequency (sample rate =  $10 \times$  mid-band frequency). Damage cycles from each of the three resulting accelerations (using MATLAB<sup>TM</sup> 'rainflow') and the corresponding total damage was calculated using the damage coefficient  $b = 6.67$ .

Results in Table 4 shows that the Damage-C is very similar to Damage-B, which is expected since the only difference is summing the ASDs followed by generation of acceleration vs. time or vice versa. Comparing Damage B vs. Damage A, it can be seen that the conservative  $R_f$  provided a Damage-B that exceeded Damage-A for each frequency band by 0.9-2.1dB (last column). No damage value is provided for the first row because there was no previous mid-frequency to generate sine-tone-equivalent ASD. Previous numerical simulations showed that the accurate  $R_f$  needs to account for the sine/original ASD rms ratio. The relevant sine/original ASD rms ratios for each octave band are also presented in Table 4, ranging from 1.03 to 2.34. Note that  $R_f$  can be as high as 1.4 (vs. conservative limit of 1.0 used in this simulation), which would result in a 3dB lower estimate of the equivalent ASD. Therefore a 0.9-2.1dB overestimation is to be expected unless a more appropriate  $R_f$  is used, which was therefore examined in future simulations.

**Table 4** Results of Damage Simulation for Each Octave Band (Using  $R_f = 1.0$ )

| Mid-band Frequency (Hz) | Duration of Dwell (min) | Sine-tone Peak (g) | Original ASD ( $g^2/Hz$ ) | Sine-tone Equivalent ASD | Damage-A (sinetone + Original ASD) | Damage-B (Equivalent ASD + original ASD) | Damage-C (Equivalent ASD + Original ASD) | Sine/ASD rms ratio | dB ratio of DamageB vs. DamageA |
|-------------------------|-------------------------|--------------------|---------------------------|--------------------------|------------------------------------|------------------------------------------|------------------------------------------|--------------------|---------------------------------|
| 1050                    | 1.03E+00                | 1.54E+00           | 2.47E-03                  | 1.44E-02                 |                                    |                                          |                                          |                    |                                 |
| 1158                    | 1.13E+00                | 1.63E+00           | 2.41E-03                  | 1.46E-02                 | 8.30E+15                           | 4.16E+16                                 | 3.85E+16                                 | 2.18               | 2.10                            |
| 1276                    | 1.25E+00                | 1.73E+00           | 2.36E-03                  | 1.49E-02                 | 1.19E+16                           | 6.09E+16                                 | 5.64E+16                                 | 2.23               | 2.12                            |
| 1407                    | 1.37E+00                | 1.84E+00           | 2.31E-03                  | 1.53E-02                 | 1.74E+16                           | 8.73E+16                                 | 9.11E+16                                 | 2.28               | 2.10                            |
| 1551                    | 1.51E+00                | 1.96E+00           | 2.25E-03                  | 1.58E-02                 | 2.41E+16                           | 8.25E+16                                 | 9.04E+16                                 | 2.34               | 1.61                            |
| 1710                    | 1.67E+00                | 1.78E+00           | 2.20E-03                  | 1.18E-02                 | 1.69E+16                           | 3.73E+16                                 | 3.92E+16                                 | 2.05               | 1.03                            |
| 1886                    | 1.84E+00                | 1.43E+00           | 2.15E-03                  | 6.89E-03                 | 7.18E+15                           | 9.73E+15                                 | 1.09E+16                                 | 1.59               | 0.40                            |
| 2079                    | 2.03E+00                | 1.04E+00           | 2.10E-03                  | 3.33E-03                 | 2.71E+15                           | 5.45E+15                                 | 5.81E+15                                 | 1.11               | 0.91                            |
| 2292                    | 2.24E+00                | 1.00E+00           | 2.05E-03                  | 2.78E-03                 | 2.86E+15                           | 5.68E+15                                 | 5.72E+15                                 | 1.03               | 0.89                            |
| 2527                    | 2.47E+00                | 1.03E+00           | 2.01E-03                  | 2.70E-03                 | 3.99E+15                           | 1.18E+16                                 | 1.17E+16                                 | 1.03               | 1.41                            |
| 2786                    | 2.72E+00                | 1.37E+00           | 1.96E-03                  | 4.27E-03                 | 9.84E+15                           | 4.20E+16                                 | 3.96E+16                                 | 1.31               | 1.89                            |
| 3072                    | 3.00E+00                | 1.73E+00           | 1.92E-03                  | 6.24E-03                 | 2.62E+16                           | 1.10E+17                                 | 1.08E+17                                 | 1.60               | 1.86                            |
| 3386                    | 3.31E+00                | 1.97E+00           | 1.87E-03                  | 7.30E-03                 | 4.71E+16                           | 1.82E+17                                 | 2.06E+17                                 | 1.75               | 1.76                            |

#### Simulation #5 Full-Frequency Range One Bandwidth at a Time ( $R_f = 1.4$ , non-conservative)

The previous simulation was then repeated using  $R_f = 1.4$  (upper non-conservative limit) and the results are shown in Table 5. Note that the higher  $R_f$  value (1.4 instead of 1.0) has lowered the sine-tone-equivalent ASDs by a factor of 2 (3dB) as expected. It has also resulted in a corresponding decrease in Damage-B and Damage-C relative to the same columns in Table 4. It is also possible to compare the variation between two simulations (using the same sine-tone + original ASD data) by comparing the column for Damage-A between Tables 4 and 5. Comparing Damage-B vs. Damage-A, it can now be seen that the non-conservative  $R_f$  provided a Damage-B that is consistently lower than Damage-A for each frequency band by up to -1.35dB (last column). These two simulations prove that the factor  $R_f$  appropriately captures the lower and upper limit of damage within a particular frequency band for a SDOF response centered in that frequency band. Further improvement of accuracy is explored next.

#### Simulation #6 Full-Frequency Range with One Bandwidth at a Time (Most Accurate, Band-specific $R_f$ )

Previous simulations [1] had shown how  $R_f$  varies based on the ratio of the sine-tone peak vs. grms of the original ASD. From this analysis the accurate values of  $R_f$  could be tabulated (Table 6) for values of the fatigue coefficient  $b$ . These more accurate  $R_f$  values can then be used for more accurate determination of equivalent ASD from each sine-tone. Using the look-up Table 6 below and the rms ratio of the sine-tone and random-vibe specification in each octave band, the appropriate  $R_f$  for each band was determined and used in the equivalencing equation.

**Table 5** Results of Damage Simulation for Each Octave Band (Using  $R_f = 1.4$ )

| Mid-band Frequency (Hz) | Sine-tone Peak (g) | Original ASD ( $\text{g}^2/\text{Hz}$ ) | Sine-tone Equivalent ASD ( $\text{g}^2/\text{Hz}$ ) $b = 6.67$ | Damage-A (sinetone + Original ASD) | Damage-B (Equivalent ASD + ASD) | Damage-C (Equivalent ASD - Original ASD) | dB ratio of DamageB Vs. DamageA |
|-------------------------|--------------------|-----------------------------------------|----------------------------------------------------------------|------------------------------------|---------------------------------|------------------------------------------|---------------------------------|
| 1050                    | 1.54E+00           | 2.47E-03                                | 7.35E-03                                                       |                                    |                                 |                                          |                                 |
| 1158                    | 1.63E+00           | 2.41E-03                                | 7.47E-03                                                       | 8.10E+15                           | 6.16E+15                        | 6.30E+15                                 | -0.36                           |
| 1276                    | 1.73E+00           | 2.36E-03                                | 7.62E-03                                                       | 1.18E+16                           | 8.88E+15                        | 1.02E+16                                 | -0.37                           |
| 1407                    | 1.84E+00           | 2.31E-03                                | 7.81E-03                                                       | 1.68E+16                           | 1.38E+16                        | 1.29E+16                                 | -0.17                           |
| 1551                    | 1.96E+00           | 2.25E-03                                | 8.04E-03                                                       | 2.46E+16                           | 1.70E+16                        | 1.28E+16                                 | -0.76                           |
| 1710                    | 1.78E+00           | 2.20E-03                                | 6.01E-03                                                       | 1.70E+16                           | 7.00E+15                        | 6.60E+15                                 | -1.16                           |
| 1886                    | 1.43E+00           | 2.15E-03                                | 3.52E-03                                                       | 7.54E+15                           | 2.67E+15                        | 2.78E+15                                 | -1.35                           |
| 2079                    | 1.04E+00           | 2.10E-03                                | 1.70E-03                                                       | 2.70E+15                           | 1.68E+15                        | 1.74E+15                                 | -0.62                           |
| 2292                    | 1.00E+00           | 2.05E-03                                | 1.42E-03                                                       | 2.89E+15                           | 1.95E+15                        | 1.94E+15                                 | -0.62                           |
| 2527                    | 1.03E+00           | 2.01E-03                                | 1.38E-03                                                       | 3.54E+15                           | 3.45E+15                        | 3.45E+15                                 | -0.18                           |
| 2786                    | 1.37E+00           | 1.96E-03                                | 2.18E-03                                                       | 9.53E+15                           | 1.01E+16                        | 1.08E+16                                 | 0.07                            |
| 3072                    | 1.73E+00           | 1.92E-03                                | 3.18E-03                                                       | 2.61E+16                           | 2.23E+16                        | 2.22E+16                                 | -0.20                           |
| 3386                    | 1.97E+00           | 1.87E-03                                | 3.72E-03                                                       | 5.11E+16                           | 3.57E+16                        | 3.51E+16                                 | -0.47                           |

**Table 6** Accurate Value of  $R_f$  for Various Ratio of rms Values

| Ratio          | 0 | 0.2  | 0.4  | 0.6  | 1    | 1.4  | 1.8  | 2.2  | 2.8  | 3.4  | 4    | 5    |
|----------------|---|------|------|------|------|------|------|------|------|------|------|------|
| $R_f-b = 6.67$ | 1 | 1.03 | 1.07 | 1.11 | 1.18 | 1.24 | 1.28 | 1.31 | 1.35 | 1.37 | 1.39 | 1.41 |
| $R_f-b = 4.0$  | 1 | 1.01 | 1.03 | 1.05 | 1.09 | 1.12 | 1.14 | 1.15 | 1.17 | 1.18 | 1.19 | 1.20 |

Results shown in Table 7 (last column) demonstrate that the band-specific  $R_f$  significantly reduces the differences to a maximum of 0.27dB, and significantly lower in most cases. Note that the variation in Damage-A in each of Table 5, 6 and 7 are only due to random process error (generation of acceleration vs. time from ASD) and subsequent processing. Convergence of Damage in 5 had shown how the simulation can fall-short when the duration is less than 100sec (for 1kHz mid-frequency band) because the underlying statistical basis for conversion is only valid when there is sufficient length

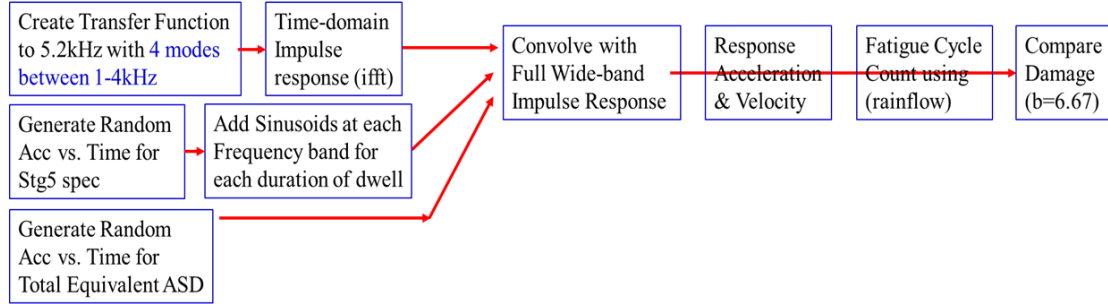
**Table 7** Results of Damage Emulation using Appropriate  $R_f$  for Each Octave Band

| Mid-band Frequency (Hz) | Sine-tone Peak (g) | Original ASD ( $\text{g}^2/\text{Hz}$ ) | Sine-tone Equivalent ASD ( $\text{g}^2/\text{Hz}$ ) $b = 6.67$ | Damage-A (sinetone + Original ASD) | Damage-B (Equivalent ASD + ASD) | Damage-C (Equivalent ASD - Original ASD) | dB ratio of DamageB Vs. DamageA |
|-------------------------|--------------------|-----------------------------------------|----------------------------------------------------------------|------------------------------------|---------------------------------|------------------------------------------|---------------------------------|
| 1050                    | 1.54E+00           | 2.47E-03                                | 8.40E-03                                                       |                                    |                                 |                                          |                                 |
| 1158                    | 1.63E+00           | 2.41E-03                                | 8.49E-03                                                       | 8.72E+15                           | 8.89E+15                        | 8.56E+15                                 | 0.03                            |
| 1276                    | 1.73E+00           | 2.36E-03                                | 8.62E-03                                                       | 1.24E+16                           | 1.30E+16                        | 1.25E+16                                 | 0.06                            |
| 1407                    | 1.84E+00           | 2.31E-03                                | 8.79E-03                                                       | 1.66E+16                           | 2.04E+16                        | 1.83E+16                                 | 0.27                            |
| 1551                    | 1.96E+00           | 2.25E-03                                | 9.03E-03                                                       | 2.53E+16                           | 2.98E+16                        | 2.62E+16                                 | 0.20                            |
| 1710                    | 1.78E+00           | 2.20E-03                                | 6.94E-03                                                       | 1.75E+16                           | 1.82E+16                        | 1.82E+16                                 | 0.05                            |
| 1886                    | 1.43E+00           | 2.15E-03                                | 4.34E-03                                                       | 7.52E+15                           | 8.52E+15                        | 7.92E+15                                 | 0.16                            |
| 2079                    | 1.04E+00           | 2.10E-03                                | 2.31E-03                                                       | 2.85E+15                           | 3.05E+15                        | 3.02E+15                                 | 0.09                            |
| 2292                    | 1.00E+00           | 2.05E-03                                | 2.05E-03                                                       | 2.89E+15                           | 3.02E+15                        | 3.08E+15                                 | 0.01                            |
| 2527                    | 1.03E+00           | 2.01E-03                                | 1.92E-03                                                       | 3.82E+15                           | 4.12E+15                        | 4.12E+15                                 | 0.07                            |
| 2786                    | 1.37E+00           | 1.96E-03                                | 2.84E-03                                                       | 9.77E+15                           | 1.12E+16                        | 1.26E+16                                 | 0.16                            |
| 3072                    | 1.73E+00           | 1.92E-03                                | 3.92E-03                                                       | 2.62E+16                           | 2.79E+16                        | 3.00E+16                                 | 0.08                            |
| 3386                    | 1.97E+00           | 1.87E-03                                | 4.47E-03                                                       | 5.08E+16                           | 5.31E+16                        | 5.59E+16                                 | 0.06                            |

of data. These two statistical issues (generation of random signals from an ASD and sufficient data to fit the Rayleigh distribution of peaks) are therefore responsible for any remaining error  $<0.3\text{dB}$ .

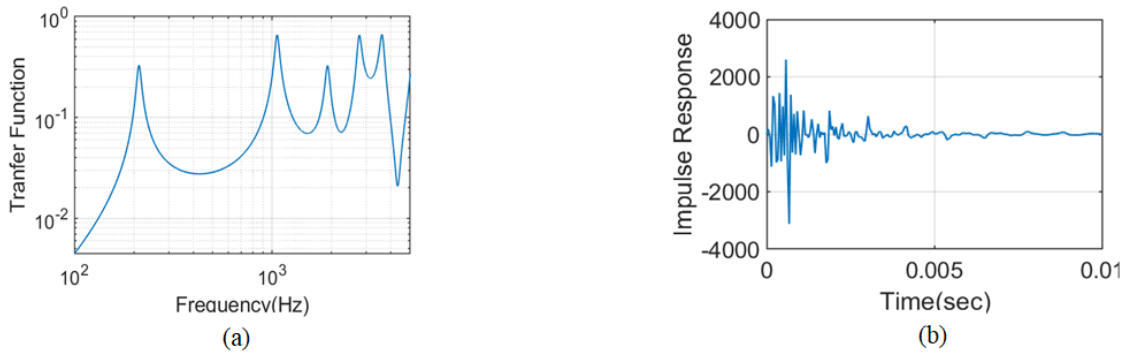
#### Simulation #7 Full-Simulation Concurrent Over Entire Frequency Band (1-4kHz)

Figure 7 shows the full-up simulation process and associated steps where all the frequency bands are occurring concurrently. Time-domain inputs are generated for each input set (original ASD, sine-tone at each frequency, equivalent ASDs). The sine-tone varies in frequency and amplitude for each dwell duration (based on the octave bands) and have to be added to the random signal at each point in time. The signals are convolved with the transient impulse response for the modes of interest (or sum of impulse functions for various modes) to obtain corresponding time-domain responses. The resulting response signal is processed using MATLAB<sup>TM</sup> ‘rainflow’ to count the number of fatigue cycles. The number and amplitude of fatigue cycles are accumulated into a scalar *damage* parameter using Miner’s rule and the fatigue exponent of  $b = 6.67$  as discussed earlier.



**Fig. 7** Sequence of steps in the full concurrent (for all bands) simulation

Equivalent vibre specs (includes both random-vibe and sine-sweep per [Eq (1)]) were first calculated using different values of  $R_f$  (1.0 for conservative and 1.17 for closer approximation). The simulation evaluates the damage from the different inputs using the sequence of steps shown in Figure 7. In order to simulate the response of the component to inputs, transfer function for an arbitrary system was assumed. The transfer function used the same damping constant ( $\xi = 0.05$ ) as the equivalencing calculations for consistency. The transfer function shown in Figure 8a shows four resonance peaks between 1 and 4kHz. The associated time-domain Impulse response is shown in Figure 8b.



**Fig. 8** Transfer function (a) and associated impulse function (b)

The three time-domain inputs that were convolved with the impulse function (Figure 8b) were the following:

- Original random-vibe signal added to the sine-sweep signal.
- Time-domain signal corresponding to the Safe Equivalent (very conservative).
- Time-domain signal corresponding to the Closer Equivalent (less conservative) ASD.

Each of the three signals were convolved with the impulse function (Figure 8b) to obtain the response. The response was then subject to a ‘rainflow’ cycle counting followed by determination of total damage using the fatigue coefficient ( $b = 6.67$ ). Results are shown in Table 8 (columns 2 and 3). The damage from the equivalent ASDs is conservative by approximately 3dB due to the use of conservative values of  $R_f = 1.0$  and  $R_f = 1.17$ .

The response and consequent damage were then calculated using a damping factor  $\xi = 0.01$  (instead of 0.05) and the results are shown in the last two columns in Table 8. Lower damping increases the response and hence the consequent damage. More importantly, the effect of each response frequency is more contained near that frequency due to the lower damping. The reduced impact of one band over adjacent bands significantly decreases the variation in damage. The overly conservative damage values from the equivalent ASDs can also be attributed to the low sampling rate ( $\approx 20000\text{Hz}$ ) relative to the highest frequency of the transfer function in Figure 8a.

**Table 8** Comparison of Damage from the Three Simulations, Variation of Damping

| Type of Input                      | Damage ( $\xi = 0.05$ ) | dB variation | Damage ( $\xi = 0.01$ ) | dB variation |
|------------------------------------|-------------------------|--------------|-------------------------|--------------|
| Sinetone+Stg5 ASD                  | 8.11E+36                | 0            | 1.84E+38                | 0            |
| Equivalent Random ( $R_f = 1.17$ ) | 5.73E+37                | 2.5          | 3.42E+38                | 0.8          |
| Equivalent Random ( $R_f = 1.0$ )  | 1.32E+38                | 3.6          | 8.97E+38                | 2            |

## Conclusion

Converting sine-sweep test specifications to equivalent random vibration ASDs and vice-versa are necessary for test execution as well as for evaluating consequences of various test in terms of imparted damage. Detailed process for converting a concurrent sine-sweep and random vibration into an equivalent random vibration test was outlined in a previous IMAC paper [2]. This paper presented the associated theoretical approximations and provided uncertainty quantifications associated with each step in the process. A sequence of numerical simulations with progressively greater complexity were undertaken to capture the gradual reduction of uncertainty based on greater understanding and quantification of the contributing factors.

## References

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## Chapter 7

# A Probabilistic Reasoner Based on Bayes Risk for Damage Detection in Structural Systems

David Najera-Flores, Justin Jacobs, D. Dane Quinn, Michael D. Todd, and Anthony Garland

**Abstract** Structural health monitoring (SHM) systems are used to inform operation of structural systems subject to loads and environments that may affect their integrity. SHM systems rely on continuous monitoring of the structure to determine its health state. These systems are often coupled with a model of the deployed structure to determine the consequences of changes in the system by forecasting the response to future states. These models, which may be thought of as digital twins, need to be updated to reflect the latest state of the structural system. This work makes use of an uncertainty-aware machine learning model that enforces distance preservation of the original input space to determine deviations from the training data input space distributions. This workflow enables domain shift detection to determine whether damage is present in the structure. The uncertainty metrics generated by this network are then used in a Bayes risk framework to design an optimal damage detector given cost and risk considerations. The approach is demonstrated on a computational example with simulated damage.

**Keywords** Machine learning · Nonlinear structural dynamics · Structural health monitoring · Damage detection · Data-driven · Structure-preserving

## Introduction

A structural health monitoring (SHM) system should enable safe and reliable operation of the monitored system by providing actionable information to stakeholders so they can perform informed decisions about the structure's maintenance. This task involves the mapping from data to knowledge via an inference process. This inference process is also known as *reasoning*. More precisely, a probabilistic reasoner takes into account the various probability distributions from uncertain events to propose a course of action given the cost of decision consequences (risk). This task implicitly involves performing a diagnosis or detection of an underlying change in the system via indirect observable symptoms [1]. The work by the authors in [2] introduced a strategy for domain shift detection based on symptoms related to the dynamic response of a system. That work presented a method based on a distance-preserving, uncertainty-aware machine learning model (ML) that acted as a digital twin. The proposed model was able to ingest data from an external source (such as sensors) and compute the potential energy

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David Najera-Flores  
ATA Engineering, Inc., San Diego, CA 92128  
e-mail: [dnajera@ata-e.com](mailto:dnajera@ata-e.com)

Justin Jacobs  
Oak Ridge National Laboratory, Oak Ridge, TN 37830  
e-mail: [jacobsjw@ornl.gov](mailto:jacobsjw@ornl.gov)

D. Dane Quinn  
Department of Mechanical Engineering, The University of Akron, Akron, OH 44325  
e-mail: [quinn@uakron.edu](mailto:quinn@uakron.edu)

Michael D. Todd  
Department of Structural Engineering, University of California San Diego, La Jolla, CA 92093  
e-mail: [mdtodd@ucsd.edu](mailto:mdtodd@ucsd.edu)

Anthony Garland  
Sandia National Laboratories, Albuquerque, NM 87123  
e-mail: [agarlan@sandia.gov](mailto:agarlan@sandia.gov)

and the damping of the underlying system, as well as the corresponding predictive variance. The work in [2] demonstrated how a Wald-type hypothesis test based on the order statistics of the distribution of the logarithm of the variance over time was useful for robust domain shift detection. The present work builds upon this past work to demonstrate how this workflow can be used to provide actionable information based on the desired sensitivity and specificity of the detector. The approach is demonstrated on a computational example with simulated damage. Section provides technical background, Section presents a computational example with simulated damage to demonstrate the proposed approach, and Section concludes and provides directions for future work.

## Background

This section provides a technical description of the ML model used and the detection theory principles employed including the uncertainty-aware ML model and the Bayes risk approach.

### Uncertainty-aware ML model

The model introduced in [2] is briefly summarized in this section. The neural network architectures used in this work consist of residual connections with spectral-normalized weights. This choice of architecture ensures distance-preservation as described in [3] by enforcing the lower bound of the Lipschitz constant through its residual connections, and the upper bound by using spectral normalization. This construction represents a bi-Lipschitz constraint that prevents disproportional contraction (also known as feature collapse) or expansion of the input space when it is transformed to the hidden space [3]. There are two separate networks and they both follow the exact same architecture. The input to the network is either  $\mathbf{z}_\beta$  or  $\dot{\mathbf{z}}_\beta$ , which are  $N_\beta$ -dimensional vectors, where  $N_\beta$  is the number of degrees-of-freedom in the boundary of the isolated region. The input layer maps the input to the hidden space of dimension  $d$  with a linear dense layer. The next  $L$  hidden layers follow a residual architecture where the input is added to the output after each dense layer. A swish activation function is used at each dense layer. The output layer uses a Gaussian process (GP) with kernel matrix,  $\mathbf{K}_{\text{GP}}$ . The hidden representation,  $\mathbf{h}_i \in \mathbb{R}^{d_{L-1} \times 1}$ , of the previous penultimate layer of dimension  $d_{L-1}$  serves as input to the GP layer.

As the output Gaussian process is intractable, a low-rank approximation of the kernel matrix is applied via

$$\mathbf{K}_{\text{GP}} = \Phi \Phi^T, \quad (1)$$

where

$$\Phi_i = \sqrt{2\sigma^2/d_L} \cos(-\mathbf{W}_L \mathbf{h}_i + \mathbf{b}_L), \quad (2)$$

are random features that form a random sampling of a square-integrable basis [4] for the output layer. Here  $\mathbf{W}_L \in \mathbb{R}^{d_L \times d_L}$  is a fixed weight matrix with entries sampled independent and identically distributed (i.i.d) from a standard normal distribution,  $\mathbf{b}_L \in \mathbb{R}^{d_L \times 1}$  is a fixed bias vector with entries sampled i.i.d. from a uniform distribution in the range  $(0, 2\pi)$ , and  $\sigma^2$  is the GP kernel variance and is set to 1.0.

The output of the SNGP is then obtained as

$$g(h_i) = \Phi_i^T \mathbf{u} + c, \quad (3)$$

where  $\mathbf{u} \in \mathbb{R}^{d_L \times 1}$  is the vector of learnable weights in the last layer and is initialized by randomly sampling from a zero-mean normal distribution with variance  $\tau^2 = 3.0$  (tuned manually for numerical stability) and  $c \in \mathbb{R}^{1 \times 1}$  is the learnable constant bias. The output  $g(h_i) \in \mathbb{R}^{1 \times 1}$  is a scalar field corresponding to either the nonlinear potential  $V_{\text{nl}}(\mathbf{z}_i)$  or nonlinear damping  $\mathcal{I}_{\text{nl}}(\dot{\mathbf{z}}_i)$ .

To obtain the prediction variance, we need to construct the model Hessian matrix,  $\mathbf{H} \in \mathbb{R}^{d_L \times d_L}$ , of the log posterior likelihood at the maximum a posteriori (MAP) estimate. This MAP estimate is induced as the Gaussian process posterior approximated in (1) and (2). Per [3], the model Hessian is defined as

$$\mathbf{H} = \sum_{i=1}^{N_{\text{train}}} \Phi_i \Phi_i^T + \mathbf{I}, \quad (4)$$

where  $N_{\text{train}}$  is the number of training examples. The Hessian matrix corresponds to the precision matrix, or the inverse of the covariance matrix  $\Sigma \in \mathbb{R}^{d_L \times d_L}$ . Hence, the prediction variance of a new example can be computed as

$$\text{var}(\mathbf{z}_k) = \Phi_k^T \Sigma \Phi_k \quad (5)$$

where  $\Sigma$  is fixed after training but  $\Phi_k$  is a function of the new input  $\mathbf{z}_k$ . All of the code was implemented using the Python packages Jax [5] and Flax [6].

### Damage detector definition

The approach taken in this work is based on statistical detection theory [7]. However, unlike classical detection theory which deals with known signals, the present problem deals with unknown processes, e.g., cases in which the underlying statistical process is not completely known *a priori*. In this work, the probability density function and its corresponding parameters are unknown since they are being generated by a neural network. This statistical process corresponds to the predictive variance associated with the damping and potential energy of the structural system. These signals are used as observable symptoms in lieu of direct observation of damage. To this end, the problem statement in this work consists of the following simple binary hypothesis test:

$$\begin{aligned} H_0 &: \text{damage is not present} \\ H_1 &: \text{damage is present} \end{aligned} \quad (6)$$

Following the hypothesis test framework developed in [2], the two hypotheses rely on the statistical dispersion of the signal being considered as

$$\begin{aligned} H_0 &: \text{IQR}_X - \text{IQR}_Z = 0 \\ H_1 &: \text{IQR}_X - \text{IQR}_Z \neq 0, \end{aligned} \quad (7)$$

where IQR is defined as the interquartile range (IQR) of the random signals  $X$  and  $Z$ , which correspond to a vector of unordered samples. This vector is defined as the logarithm of the predictive neural network variance within a predefined time window. In this case, the signal  $X$  corresponds to the predictive variance associated with a baseline case, which is representative of the healthy state of the structure, while  $Z$  represents other time signals to be tested.

An empirical distribution of the random signals,  $X$  and  $Z$ , is obtained through a bootstrap of  $K$  samples drawn within a predefined time window. Consistent with [2], a Wald-type test is carried out explicitly by a Welch's  $t$ -test defined as

$$T = \frac{\bar{X} - \bar{Z}}{\sqrt{S_X^2 + S_Z^2}}. \quad (8)$$

The test statistic (8) then follows a  $t$ -distribution with degrees of freedom

$$\text{df} = \frac{\left(\frac{S_Z^2}{K} + \frac{S_X^2}{K}\right)^2}{\frac{1}{K-1} \left(\frac{S_Z^2}{K}\right)^2 + \frac{1}{K-1} \left(\frac{S_X^2}{K}\right)^2}. \quad (9)$$

The two-tailed p-value associated with the test statistic in (8) can then be computed as

$$p = 2(1 - F(T)) \quad (10)$$

where  $F(T)$  represents the cumulative density function (CDF) of the test statistic. The hypothesis test can therefore be expressed as

$$\begin{aligned} p &\geq \alpha : \text{fail to reject } H_0 \\ p &< \alpha : \text{we reject } H_0 \end{aligned} \quad (11)$$

where  $\alpha$  represents the significance level of the hypothesis test, also interpreted here as the probability of false alarm ( $P_{FA}$ ). In other words, the significance level  $\alpha$  defines the probability  $P_{FA} = P(H_1|H_0)$ , or the probability of deciding that the structure is damaged when it is not, also known as a type I error.

Furthermore, this binary hypothesis test also allows us to compute the associated power of the test (denoted as  $\beta$ ) which defines the probability that the test correctly rejects  $H_0$  when  $H_1$  is true. This correct inference is known as the probability of detection,  $P_D = P(H_1|H_1)$ . For most detectors, there exists a trade-off between the probability of false alarms and the probability of detection. This trade-off is the same as the trade-off between the specificity and sensitivity of the test (or the probability of false alarms vs. misses). Therefore, designing a detector is a task that must consider this trade-off in a systematic way. One approach to accomplish this is to rely on the concept of Bayes risk [7],  $\mathcal{R}(\theta)$ , which combines the weighted likelihood of each possible decision as

$$\mathcal{R}(\theta) = E(C, \theta) = \sum_{i=0}^1 \sum_{j=0}^1 C_{ij} P(H_i|H_j, \theta) P(H_j) \quad (12)$$



where the coefficients  $C_{ij}$  represent the cost associated with deciding  $H_i$  when  $H_j$  is true,  $\theta$  are the fixed choice parameters associated with the detector or decision function, the likelihood terms  $P(H_i|H_j, \theta)$  can be computed from the hypothesis test formulation as described above, and the prior terms  $P(H_j)$  can be defined based on prior knowledge (for example, from historical data). The Bayes risk can be interpreted as the smallest integrated risk over all decision functions [8]. These decision functions are parameterized by  $\theta$  which enables minimization of the risk function by considering all the possible  $\theta$ .

## Analysis

This section will demonstrate the approach described previously by applying the proposed workflow to a structural rod with isolated deviatoric behavior representative of damage.

### Problem description

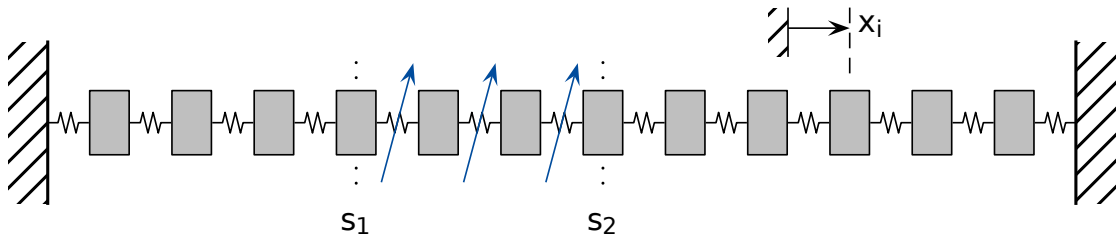
As described previously, a Wald-type hypothesis test was used to detect damage in the rod structure described in [9] and shown in Figure 1. This example is described in detail in the original paper but a general description is provided here. The rod problem was defined using the isolated formulation introduced in [10, 11] where a structure was decomposed into two regions  $C_1$  and  $C_2$ , where the equations of motion within  $C_1$  are assumed to be linear and known while region  $C_2$  contains localized nonlinearities such as joints or damaged regions. Based on this formulation and following the work in [10], the equation of motion can be decomposed as

$$\begin{aligned} \mathbf{M} \ddot{\mathbf{w}} + \mathbf{C} \dot{\mathbf{w}} + \mathbf{K} \mathbf{w} &= \mathbf{f}(t) - \mathbf{Q}_\star, \\ \mathbf{M}_2 \ddot{\mathbf{z}} + \mathbf{C}_2 \dot{\mathbf{z}} + \mathbf{K}_2 \mathbf{z} + \mathbf{N}(\mathbf{y} + \mathbf{z}) &= \mathbf{0}, \\ \mathbf{Q}_\star &= [\mathbf{0}_c \quad \mathbf{Q} \quad \mathbf{0}_2]^T, \\ \mathbf{Q} &\equiv \mathbf{M}_{\alpha\beta} \ddot{\mathbf{z}}_\beta + \mathbf{C}_{\alpha\beta} \dot{\mathbf{z}}_\beta + \mathbf{K}_{\alpha\beta} \mathbf{z}_\beta. \end{aligned} \quad (13)$$

where  $\mathbf{N} \in \mathbb{R}^{N_2}$  represents the vector for isolated nonlinear forces,  $\mathbf{M}, \mathbf{K}, \mathbf{C}$  represent the known mass, stiffness, and damping matrices, respectively while  $\mathbf{w} \in \mathbb{R}^{(N_1+N_2)}$  is defined as a mixed displacement vector

$$\mathbf{w} \equiv [\mathbf{x}_1 \quad \mathbf{y}]^T = [\mathbf{x}_c \quad \mathbf{x}_\alpha \quad \mathbf{y}_\beta \quad \mathbf{y}_n]^T, \quad (14)$$

combining  $\mathbf{x}_1 \in \mathbb{R}^{N_1}$ , the *exact* response of the system in  $C_1$ , with  $\mathbf{y} \in \mathbb{R}^{N_2}$ , the response of the ideal system, that is, the system in the absence of the nonlinearities, but subject to the force  $\mathbf{Q}$  applied at the boundary of the two regions, and referred to as the deviatoric force. This deviatoric force is typically unknown because the exact nature of the nonlinearity in the isolated region is unknown and therefore needs to be learned. Finally, within  $C_2$  the deviatoric response is defined as  $\mathbf{z} \equiv \mathbf{x}_2 - \mathbf{y} \in \mathbb{R}^{N_2}$ . These coordinates are further decomposed, so that the subscript  $c$  represents the DOFs in  $C_1$  that are not coupled to the interface DOFs, while  $\alpha$  represents the DOFs in  $C_1$  that are coupled to the interface. Likewise  $\beta$  represents the DOFs in  $C_2$  that are coupled to the interface (i.e., the only DOFs in  $C_2$  that are observable), and  $n$  represents the DOFs in  $C_2$  that are not coupled to the interface (i.e., not observable).



**Fig. 1** Exemplar rod discretized into a series of masses, springs, and dampers with an isolated nonlinear region.

For the rod structure, proportional damping was defined as  $\mathbf{C} \equiv \xi \mathbf{K}$ . The nonlinear region  $C_2$  is located in the interval  $(s_1, s_2) = (0.25, 0.35)$ , so that the isolated region exists between elements  $\ell \equiv 16$  and  $r \equiv 22$ . The deviatoric force across the isolated region is identified as

$$\delta Q \equiv Q_{23} - Q_{15}, \quad (15)$$

The initial conditions of the system are specified in terms of the modal displacements and velocities  $(\mathbf{q}(0), \dot{\mathbf{q}}(0))$ , where

$$\mathbf{x}(0) = \Phi \mathbf{q}(0), \quad \dot{\mathbf{x}}(0) = \Phi \dot{\mathbf{q}}(0), \quad (16)$$

where  $\Phi$  is the modal transformation matrix.

The nonlinearities in the isolated region are assumed to contain linear mistuning and cubic nonlinearities in terms of both stiffness and damping. Finally, hysteretic nonlinear damping is also present between the elements within  $\mathcal{C}_2$ , so that the nonlinear force is represented as

$$\begin{aligned} f_{nl,i} = & \kappa_i (x_{i+1} - x_i) + \chi_i (\dot{x}_{i+1} - \dot{x}_i) \\ & + \eta_i (x_{i+1} - x_i)^3 \\ & + \zeta_i (\dot{x}_{i+1} - \dot{x}_i)^3 + g_{hys,i}, \end{aligned} \quad (17)$$

where the hysteretic force is defined as

$$\begin{aligned} g_{hys,i} = & k_h ((x_{i+1} - x_i) - r_i), \\ \dot{r}_i = & \epsilon_i \tan \left( \frac{k_h}{\mu} ((x_{i+1} - x_i) - r_i) \right), \end{aligned} \quad (18)$$

following the regularized formulation presented in [9].

Unless noted otherwise the parameters of the nonlinear region are chosen as

$$\begin{aligned} \kappa_i = 0, \quad \chi_i = 0, \quad \eta_i = \rho_\eta \cdot 2.621 \times 10^5, \\ \zeta_i = 2.621 \times 10^3, \quad k_{h,i} = \rho_{k_h} \cdot 6.50 \times 10^1, \\ \mu_i = 1.00 \times 10^{-1}, \quad \epsilon_i = 1.00 \times 10^{-2}, \end{aligned} \quad (19)$$

with  $\xi = 10^{-4}$ , and  $\rho_\eta$  and  $\rho_{k_h}$  represent the level of cubic nonlinearity and hysteresis, respectively. For the data generated for training the network,  $\rho_\eta = 1.0$ , and  $\rho_{k_h} = 0.0$ . Damage was simulated by adjusting these two coefficients, as summarized in Table 1.

**Table 1** Domain shift cases. For the gradual stiffness increase, the max value is provided. The initial condition (IC) coefficients represent the contribution of each mode shape to the initial deformation.

| Case name                 | $\rho_\eta$ | $\rho_{k_h}$ | IC            |
|---------------------------|-------------|--------------|---------------|
| Baseline 0 (train)        | 1           | 0            | (1.0,0.5,2.0) |
| Baseline 1 (test)         | 1           | 0            | (0.5,1.0,0.5) |
| Baseline 2 (test)         | 1           | 0            | (1.0,1.0,1.0) |
| Hysteresis Lvl. 1         | 1           | 1            | (1.0,0.5,2.0) |
| Hysteresis Lvl. 2         | 1           | 10           | (1.0,0.5,2.0) |
| Hysteresis Lvl. 3         | 1           | 20           | (1.0,0.5,2.0) |
| Hysteresis Lvl. 4         | 1           | 50           | (1.0,0.5,2.0) |
| Hysteresis Lvl. 5         | 1           | 100          | (1.0,0.5,2.0) |
| Cubic Stiff. Lvl. 1       | 2           | 0            | (1.0,0.5,2.0) |
| Cubic Stiff. Lvl. 2       | 5           | 0            | (1.0,0.5,2.0) |
| Cubic Stiff. Lvl. 3       | 20          | 0            | (1.0,0.5,2.0) |
| Cubic Stiff. Lvl. 4       | 100         | 0            | (1.0,0.5,2.0) |
| Grad. Cubic Stiff. Lvl. 1 | 5           | 0            | (1.0,0.5,2.0) |
| Grad. Cubic Stiff. Lvl. 2 | 20          | 0            | (1.0,0.5,2.0) |
| Grad. Cubic Stiff. Lvl. 3 | 100         | 0            | (1.0,0.5,2.0) |

In addition, 5% synthetic measurement noise was added to the input data. Previous work [12], demonstrated that these trained neural networks are robust to measurement noise even when added at every iteration during time integration. This noise was added with the intent of approximating a more realistic sensor measurement. The additive noise was modeled as

$$\mathbf{x}_\beta(t) = \bar{\mathbf{x}}_\beta(t) + \delta, \quad (20)$$

where  $\bar{x}_\beta(t)$  is the mean process and  $\delta$  is the random noise which is assumed to be i.i.d modeled as

$$\delta = \nu \cdot \sqrt{\frac{\sum_i^N \bar{x}_\beta(t_i)^2}{N-1}} \cdot \text{Normal}(0, 1), \quad (21)$$

where  $\nu = 0.05$  and it represents the noise factor which can also be interpreted as the reciprocal of the signal-to-noise ratio (SNR) since it represents the ratio between the noise variance and the signal variance.

The deviatoric response inherits the same additive noise as

$$\mathbf{z}_\beta(t) = \bar{\mathbf{z}}_\beta(t) + \delta, \quad (22)$$

and similarly for the deviatoric velocities,  $\dot{\mathbf{z}}_\beta(t)$ . These are then used to compute the deviatoric forces as

$$\mathbf{Q}_* = \frac{\partial V_{\text{nl}}(\mathbf{z}_\beta)}{\partial \mathbf{z}_\beta} + \frac{\partial \mathcal{I}_{\text{nl}}(\dot{\mathbf{z}}_\beta)}{\partial \dot{\mathbf{z}}_\beta}, \quad (23)$$

where  $\mathbf{z}_\beta$  and  $\dot{\mathbf{z}}_\beta$  are the deviatoric displacements and velocities at the measured boundary (as described in more detail in [9]). The  $V_{\text{nl}}(\cdot)$  and  $\mathcal{I}_{\text{nl}}(\cdot)$  terms represent the nonlinear potential and damping functions, which are modeled with a structure-preserving, spectral-normalized neural Gaussian process (sp-SNGP) as described in [2]. Consistent with previous work, this work focuses on the study of the damping term variance predicted by sp-SNGP for domain shift detection as an indicator for damage.

### Optimizing for probability of detection

Designing an optimal detector depends on the specific objectives of the SHM system. One reasonable objective is to maximize the probability of detection or, in other words, design a highly sensitive detector (i.e., a high recall rate). We can use the receiver operating characteristic (ROC) curve to study this objective. Moreover, the ROC curve also shows the trade-off between recall and precision. In other words, a highly-sensitive test will also result in a larger false alarm rate. To study this trade-off, we consider the case where progressive damage is introduced via a gradual change to the cubic stiffness term. Referring to the nonlinearity described in Eq. 17, the cubic stiffness term is modified as

$$f_{\text{nl},i} = \eta_i(x_{i+1} - x_i)^3 + \eta_{\text{d},i}(t - t_{\text{d}})^2(x_{i+1} - x_i)^3, \quad (24)$$

where

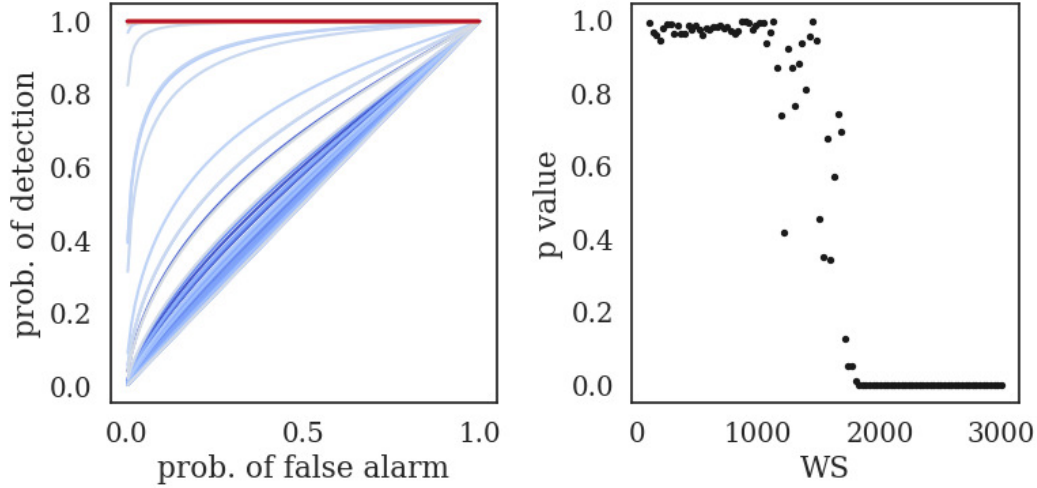
$$\begin{cases} \eta_{\text{d}} = 0.6944, & \text{if } t \geq t_{\text{d}} = 4 \\ 0 & \text{otherwise,} \end{cases} \quad (25)$$

where  $\eta$  is the cubic stiffness coefficient,  $\eta_{\text{d}}$  is the damage coefficient, and  $t_{\text{d}}$  is the time at the onset of damage. As shown, the damage growth is modeled as a quadratic function to represent a self-accelerating degradation process (i.e., the second derivative is non-zero).

Given this progressive damage model, we compute the ROC curve as a function of window size and time. We accomplish this by defining a sliding window across the time history by having consecutive windows of size  $WS$  overlap 25 time steps over the range  $t = 3$  (3000 time steps) to  $t = 4$  (4000 time steps). The following window sizes are considered:  $WS = [250, 1000, 3000, 5000]$ . Figure X shows the ROC curves for the four  $WS$  values considered, with the brightness of each line representing time evolution (darker lines representing later time). As shown, as the lines get darker, the ROC curves show an ideal detector (i.e., the area under the curve (AUC) approaches 1). This result exemplifies how damage is easier to detect as time progresses which is consistent with the self-accelerating damage evolution model. As a result, one may choose the detector that maximizes the probability of detection, which corresponds to the detector that observes data in the most damaged state. However, this behavior may be unacceptable for a realistic SHM system where decisions need to be taken in a promptly manner. To this end, Section will consider the cost associated with each decision as well as the consequence of a delay in decision making.

Given this progressive damage model, we compute the ROC curve as a function of window size and time. We accomplish this by defining a window with a starting point fixed at  $t = 4$  and variable window size. Figure 2 shows the ROC curves for 100  $WS$  values corresponding to  $WS = \{100, 129, 158 \dots 3000\}$ , with the color of each line representing time evolution (from blue to red). As shown, as the lines get more red, the ROC curves approach an ideal detector (i.e., the area under the curve (AUC) approaches 1 and the p-value approaches 0). This result exemplifies how damage is easier to detect as time progresses which is consistent with the self-accelerating damage evolution model. This effect is also evident in the IQR

histograms shown in Figure 3 where it can be seen that the distributions are hard to distinguish when small window sizes are used while they become easier to distinguish as more data are collected (also consistent with the small p-values in Figure 2). As a result, one may choose the detector that maximizes the probability of detection, which corresponds to the detector that observes data in the most damaged state. However, this behavior may be unacceptable for a realistic SHM system where decisions need to be taken in a time-critical manner. To this end, Section will consider the cost associated with each decision as well as the consequence of a delayed decision.



**Fig. 2** Left: ROC curves colored by WS value (blue to red: short to large). Right: P-value as a function of WS.

### Considering cost of wrong decisions

As described in Section, a detector that only considers probability of detection tends to favor features corresponding to an extended monitoring period. However, waiting for data may have consequences (e.g., risk of structural failure) and/or additional monitoring cost associated with data collection. For this reason, an additional term to model this penalty is added to the Bayes Risk expression as

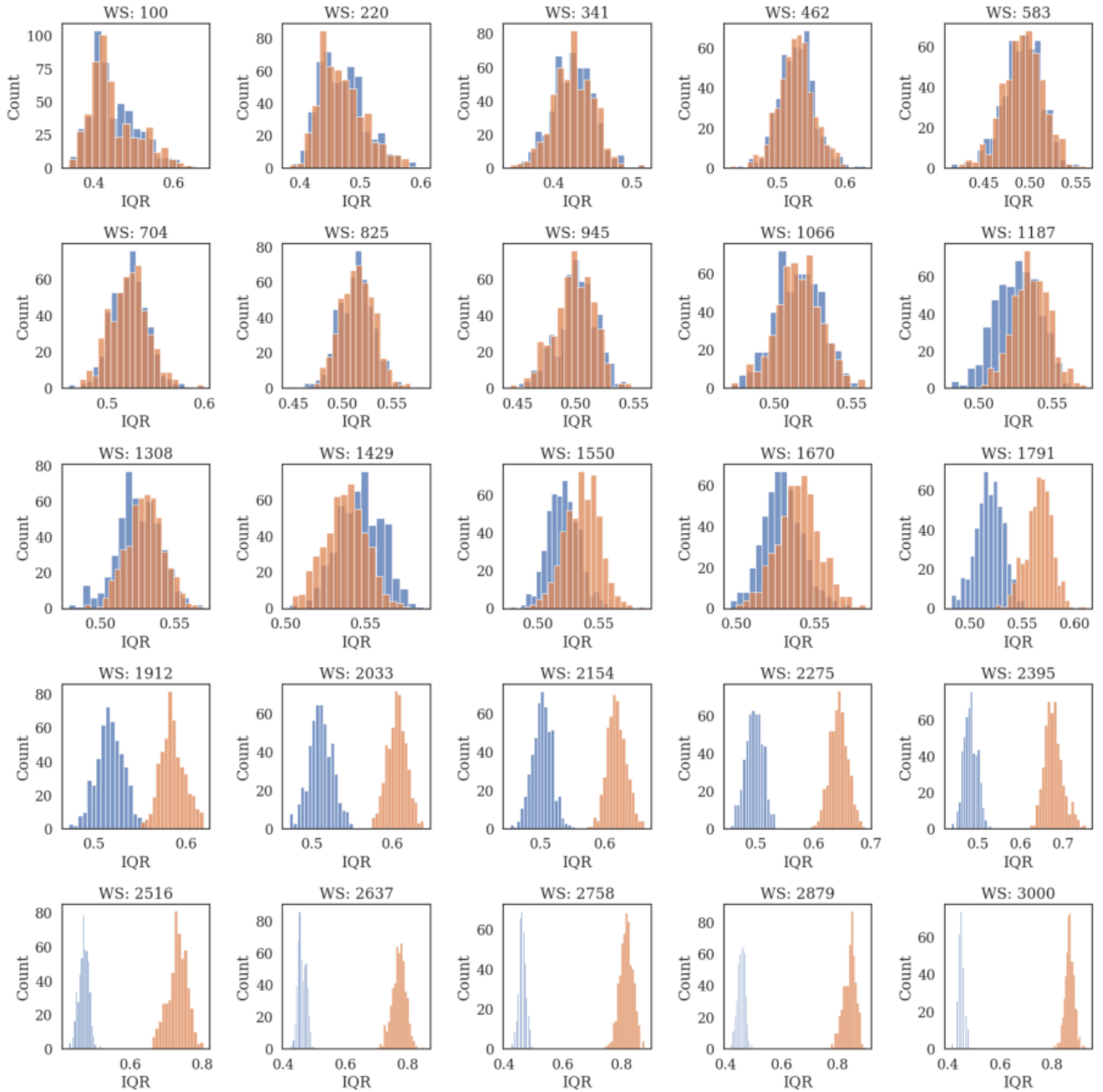
$$\mathcal{R}(\theta) = E(C, \theta) = \sum_{i=0}^1 \sum_{j=0}^1 C_{ij} P(H_i | H_j, \theta) P(H_j) + \mu \theta^2, \quad (26)$$

where  $\mu = 1.0 \times 10^{-7}$  is a normalizing coefficient. A quadratic term was chosen for the penalty term after the quadratic penalty that comes out of assuming a Gaussian prior for  $\theta$  (see ridge regression in [13]). Furthermore, normalizing the expression by the cost of a correct detection ( $C_{11}$ ), and setting the cost of normal operations to  $C_{00} = 0$ , substituting in the probability definitions, and setting  $\theta = WS$ , we obtain

$$E(C, WS) = \lambda(1 - P_D)P_{damage} + \gamma P_{FA}(1 - P_{damage}) + P_D P_{damage} + \mu WS^2, \quad (27)$$

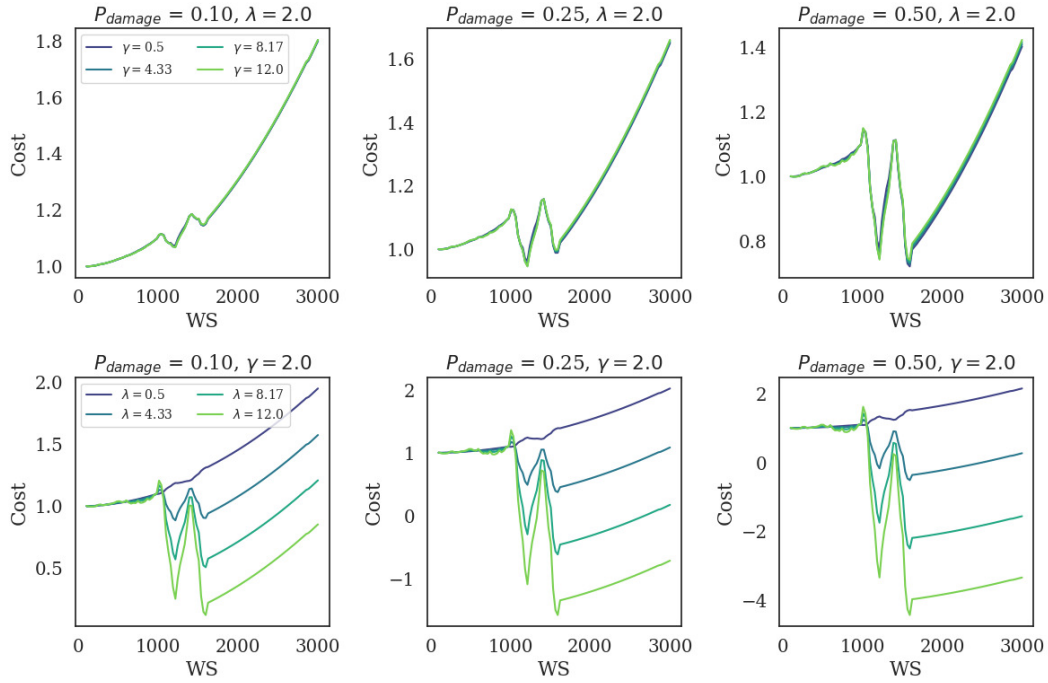
where  $\gamma = C_{10}/C_{11}$  and  $\lambda = C_{01}/C_{11}$ .

With this Bayes Risk formulation as our cost model, we can explore the effect of the cost parameters, WS, our threshold for false alarms and misses, and the prior probability. Figure 4 illustrates the effect of the prior probability of damage, the coefficients  $\lambda$  and  $\gamma$ , and WS. The cost has been normalized to be 1.0 at  $WS = 0$  for these plots for ease of visualization. The top row of plots considers the variation of  $\gamma$  which represents the ratio of the cost of a false alarm with respect to the cost of a correct detection. So  $\gamma$  essentially models the cost of unnecessary downtime due to a false alarm. For these plots,  $\lambda$  is fixed at 2. As shown in the top row of plots, the cost functions are relatively insensitive to  $\gamma$  given a fixed prior probability of damage. Most of the variation is coming from varying  $P_{damage}$ . When  $P_{damage} = 0.1$ , which means that the prior probability of damage is low (e.g., based on historical trends), the best detector is the one with the smallest WS because, as described above, there is a quadratic cost associated with WS. However, as  $P_{damage}$  increases, the minimum cost moves to the right of the x-axis, indicating that more samples are preferable, even if an additional cost is incurred, because the probability of finding damage is higher. The second row of plots shows the sensitivity to  $\lambda$  which represents the cost of



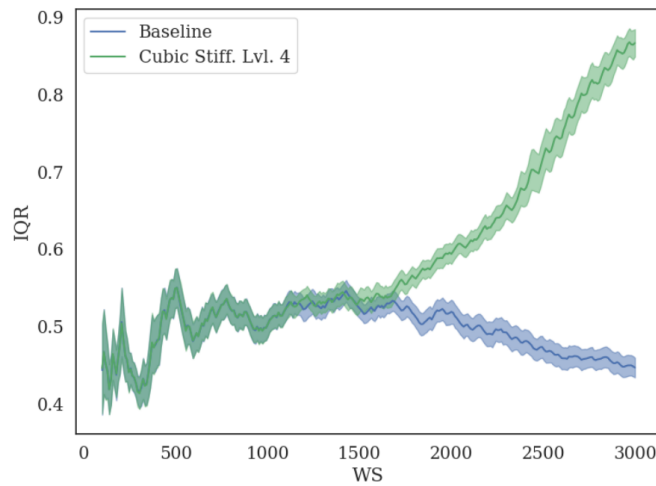
**Fig. 3** IQR histograms for varying window sizes.

failure. A large  $\lambda$  is associated with significant consequences of failure, such as it would be the case in a primary structural component. All the plots in the second row favor a non-zero WS except when  $\lambda$  is smallest. There are two valleys in all of the plots and the absolute minimum varies depending on the value of  $\lambda$ . When  $\lambda = 0.5$ , in other words the consequence of missing to detect damage is half of the cost of correctly detecting it, the optimal detector is that with the smallest WS. This scenario is clearly a very unlikely one since the cost of an error is typically higher than the cost of detecting failure before it happens. As  $\lambda$  increases, the minimum cost moves from  $WS = 0$  to the first valley and then to the second valley as  $P_{damage}$  also increases. This result is reasonable as the cost associated with the consequences of undetected failure overcome the cost of data collection to a certain extent. As we know from Section , there are diminishing returns beyond a certain WS value (i.e., the p-value plateaus in Figure 2). This result stands in contrast to the conclusion reached from Figure 2 where cost/risk was not considered and the best detector corresponded to the one that collected the most data. To understand why



**Fig. 4** Cost vs. WS as a function of prior probability of damage and  $\lambda$ .

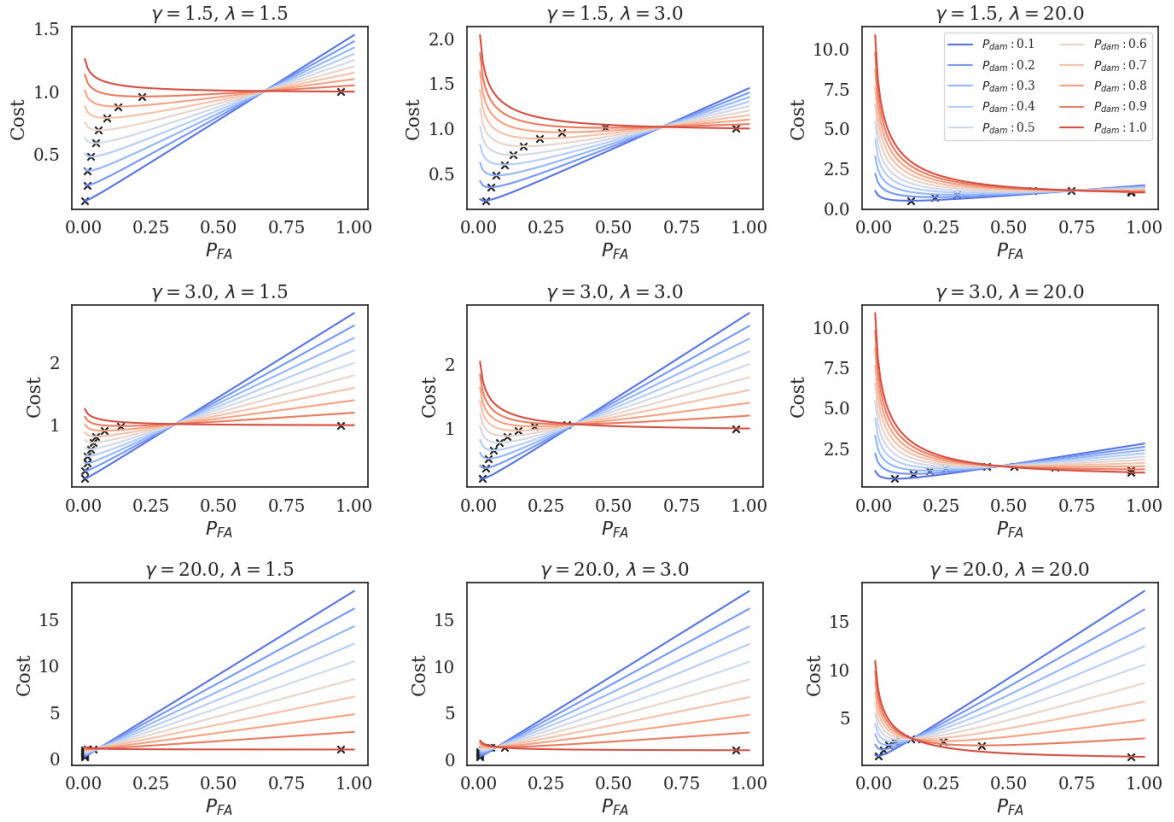
there is variability between  $WS = 1000$  and  $WS = 2000$ , we can look at the trends of the IQR distributions in Figure 5 (as a reminder, these were obtained through bootstrapping as described in Section ). As shown, the IQRs only start to deviate at around the 1000 WS mark and they exhibit an oscillatory pattern because the variances are themselves oscillatory (due to their dependence on the state space response). The first minimum in the cost function corresponds to the first evident deviation of the IQRs around  $WS = 1200$ . After that, the IQRs tend to be hard to distinguish up until the  $WS = 1600$  mark where we find the second minimum of our cost. It is expected that this transition region will vary based on the type of damage and the specific trajectories being considered so a robust detector should consider the second minimum after the oscillatory behavior has settled.



**Fig. 5** IQR mean and one standard deviation band around it for the baseline (blue) and the damaged case (green).

The same data are plotted in Figure 6 for a fixed  $WS = 1330$  to study the effect of  $\gamma$ ,  $\lambda$ , and the probability of false alarm. The value of  $\gamma$  increases as we move from top to bottom, and the value of  $\lambda$  increases as we go from left to right.



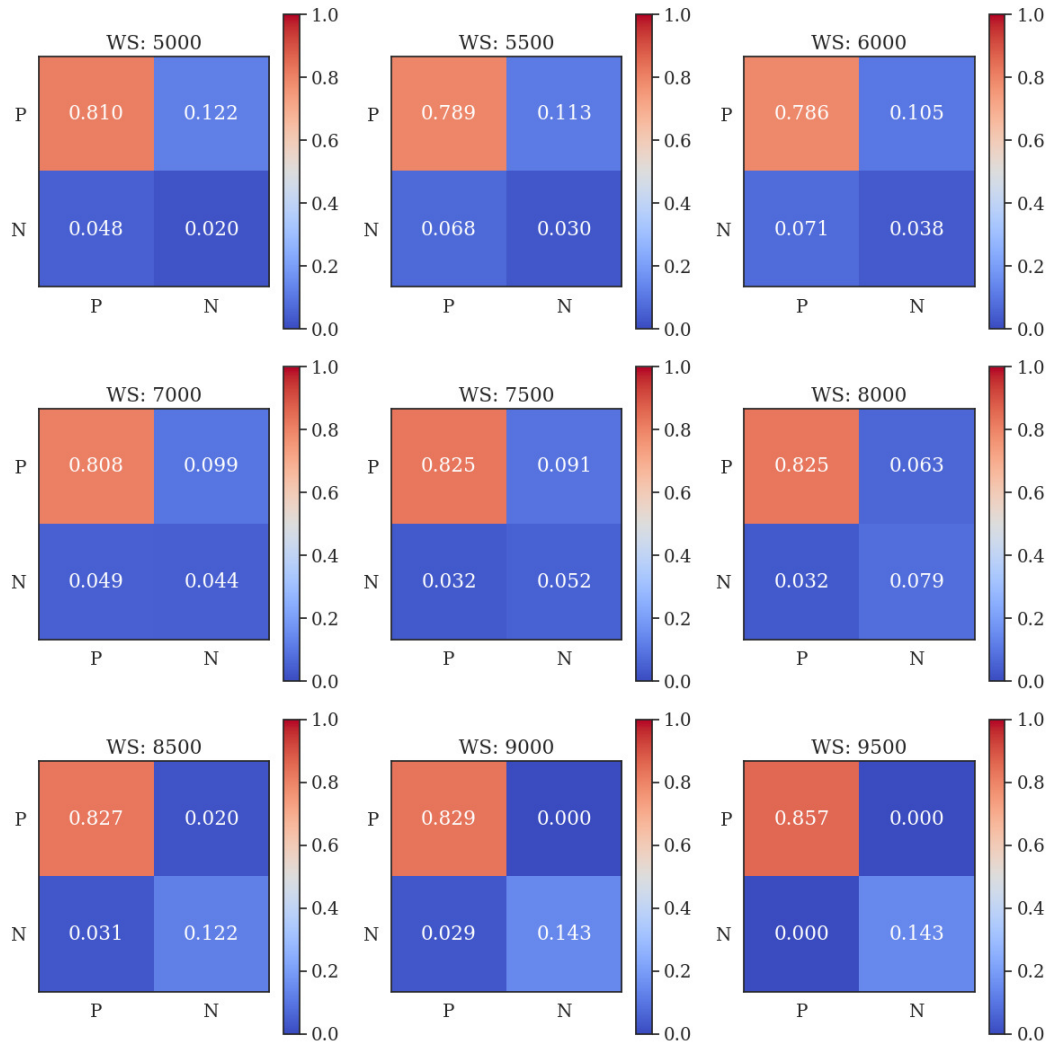


**Fig. 6** Cost vs. probability of false alarm as a function of prior probability of damage,  $\lambda$ , and  $\gamma$ .

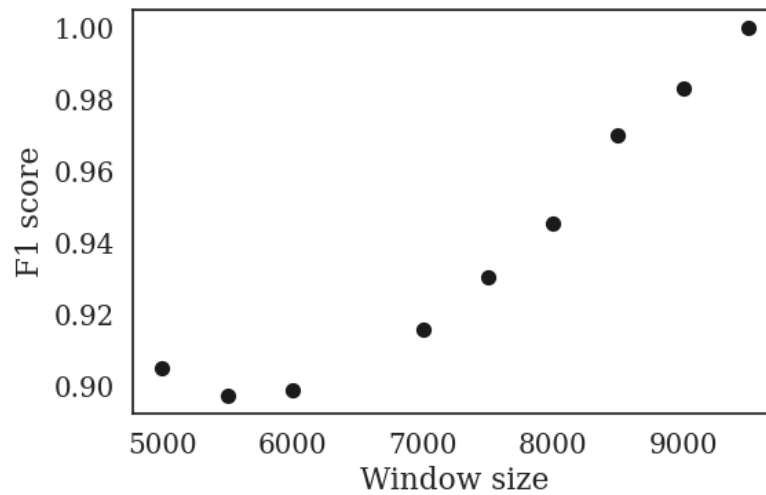
In all cases,  $\gamma > 1.0$ ,  $\lambda > 1.0$  to study the case where the cost of an error is higher than the cost of a correct detection. In all of these plots the color of the curves represents the prior probability of damage and the minimum cost of each curve is marked with  $\times$ . If we start by looking at the top plot, we can see that the minimum cost changes as we vary  $P_{\text{damage}}$ . In other words, the admissible rate of false alarms grows as  $P_{\text{damage}}$  increases. The minimums are clustered around low false alarm rates when  $\lambda$  is smaller while they quickly move to larger false alarm rates when we increase  $\lambda$ . This trend is due to the cost of an undetected failure growing. In other words, knowing that damage is likely, it is preferable to have a large rate of false alarms since undetected damage is very costly. As we move from top to bottom, the cost of unnecessary downtime increases. One extreme case is the lower left corner where  $\gamma = 20.0$ ,  $\lambda = 1.5$ . This case represents an extreme where the cost of unnecessary downtime far exceeds the cost of undetected failure (also knowing that the prior probability of damage is low). In this case, having a very low rate of false alarms is preferable. However, as the probability of damage increases, the plots on the right hand column show that even when the cost of making mistakes is high, a high rate of false alarms is still preferable. When there is a large cost associated with unnecessary downtime, the cost is proportional to the probability of false alarm, except for the case where  $P_{\text{damage}} = 1$  which is when damage is certain.

### **Detector performance when considering multiple damage types**

Sections and only considered the progressive damage case (gradual cubic stiffness change) as an exemplar. In this section we consider all of the damage cases described in Table 1. We still apply the same Wald-type hypothesis test previously described. However, the power of the test cannot be easily computed across multiple damage cases so we evaluate the detector performance by fixing  $P_{FA} = 0.05$  and generating Monte Carlo samples and then counting the true positives (TP), true negatives (TN), false positives (FP) and false negatives (FN), where TP is equivalent to  $P_D$ . The results across all cases are presented in the confusion matrices shown in Figure 7 as a function of WS. This information is further summarized by computing the F1 score as a function of WS, as shown in Figure 8. As illustrated, the F1 score improves with increased WS, leading to the same conclusion drawn in Section that the best detector is the one that collects the most data. However, as shown in Section , this result changes when the cost/risk information is considered.



**Fig. 7** Confusion matrix across all damage cases as a function of window size. Predicted condition shown across horizontal axis and actual conditions shown across vertical axis.



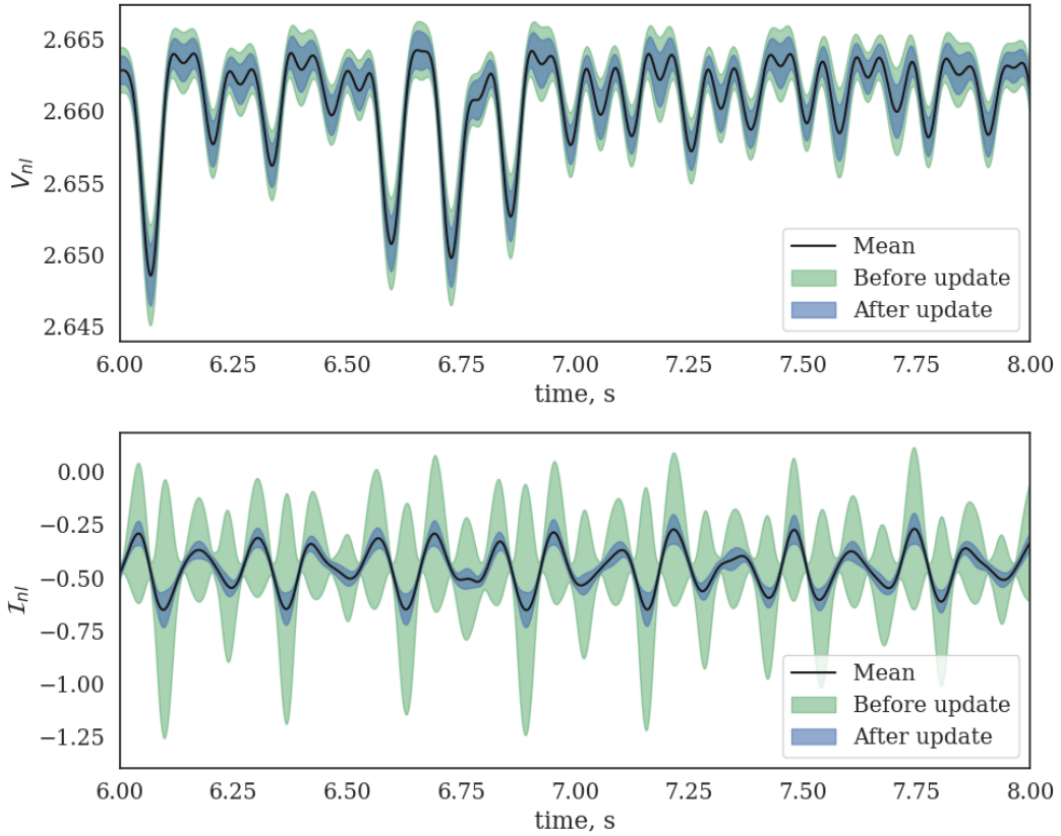
**Fig. 8** F1 score as a function of window size.



## Model updating

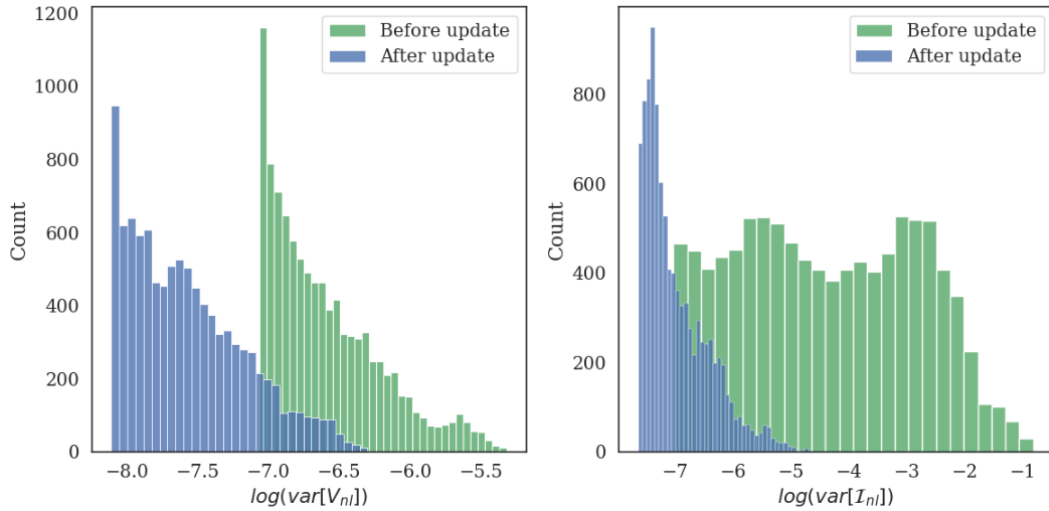
The previous section demonstrated an approach for designing a detector based on desired properties defined by cost or risk of making certain types of mistakes. Once damage or domain shift has been detected, a logical next step is to update the model to reflect the current state of the monitored structure. To demonstrate the strategy for model updating, we will focus on the example with a large change to the cubic stiffness coefficient (Cubic Stiff Lvl. 4 in [2]). This example was selected because it induced the largest error out of all the cases considered. Once damage has been detected using the approach previously presented, the existing sp-SNGP model is re-trained for a few iterations using the new data being observed. For this training, a window size of 3500 time steps from the new data being measured at the boundary of the isolated region was used to update the model. The model updates are performed by the classical backpropagation technique which involves stochastic gradient descent (SGD). The authors in [14] provide an interpretation of SGD with momentum as a posterior sampler in a Bayesian inference context. In this way, the model updating strategy employed in this work can be interpreted as a form of Bayesian inference with adaptive parameters via the ADAM optimizer.

The backpropagation algorithm was run for 10,000 epochs (it took 15 minutes on a single CPU) to update the network weights. There are two separate networks that model the nonlinear potential energy ( $V_{nl}$ ) and the nonlinear damping ( $\mathcal{I}_{nl}$ ) respectively. Figure 9 shows a close-up of the nonlinear potential energy and damping time histories and their respective predictive intervals obtained by applying different initial conditions than those used for training. As shown, the model's uncertainty is significantly reduced after the update. Figure 10 illustrates the distributions of log-variances over 10,000 time steps. As shown, the log-variance is reduced several orders of magnitude in both cases. This result is consistent with the previous findings when running sp-SNGP which are that reduced uncertainty is associated with in-distribution data. This update has effectively shifted the domain of the ML model so that the updated data is now in-distribution.



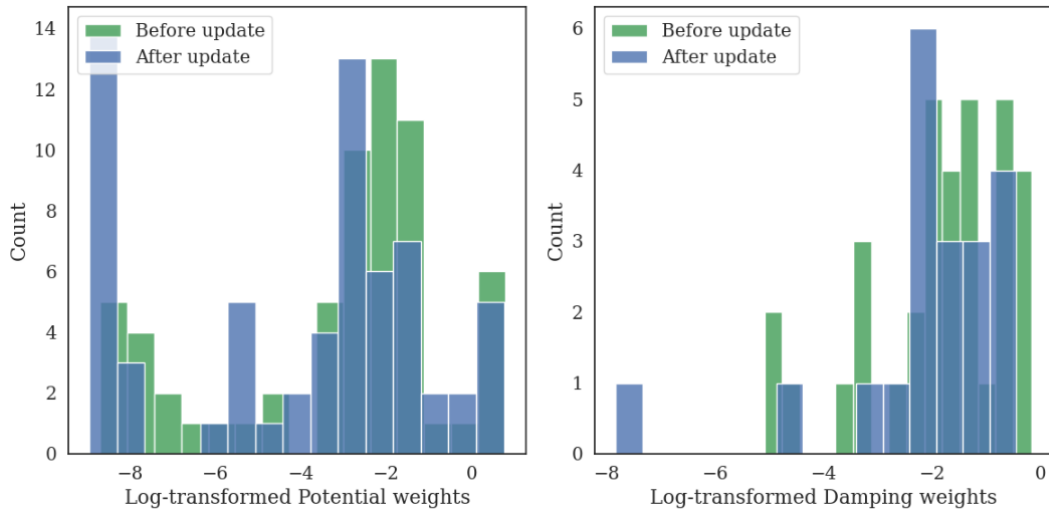
**Fig. 9** Nonlinear potential (top) and damping (bottom) time histories. One standard deviation intervals shown.

As mentioned, there are two separate networks that need to be updated. In the absence of prior knowledge about the type of nonlinearity induced by the damage, both networks would need to be updated. The distribution of non-zero log-transformed weights for both networks are presented in Figure 11 before and after the model update. As shown, the distributions before and after are fairly close to each other. To evaluate whether or not they are the same distribution, we employed a Kolmogorov-Smirnov (KS) two-sample test. The p-value for the nonlinear potential weight distributions was 0.030 while for



**Fig. 10** Histograms of log-variance of nonlinear potential (left) and damping (right) network weights.

the nonlinear damping weight distributions it was 0.626. Considering a threshold of  $p = 0.05$ , we reject the null hypothesis of the KS test that the nonlinear potential weights were drawn from the same distribution, while we fail to reject the null hypothesis for the nonlinear damping, leading to the conclusion that the two samples were drawn from different distributions. In other words, the difference between the nonlinear potential weight distributions is statistically significant while it is not for the nonlinear damping. This result suggests that the domain shift observed in the Cubic Stiff. Lvl. 4 was caused by a change to the nonlinear potential, which is consistent with the physical change (an update to the cubic stiffness term), which only contributes to the nonlinear potential function. This result can be used to infer the type of change to the system from the data in the absence of any knowledge informing the physical mechanism that is causing the damage symptoms. Future work could extend this work to fully parametric formulation that enables parameter identification.



**Fig. 11** Histograms of log-variance of nonlinear potential (left) and damping (right).

## Conclusion

This work presented a demonstration of a strategy for cost/risk-informed decision-making based on features extracted from an uncertainty-aware ML model. The ML model predictive variance was used as the basis for defining a feature to detect

damage. Based on the work presented in [2] and the discussion in the present paper, it was shown that the order statistics of the log-transformed variance of the damping time histories could be used to detect domain shift which in turn was used as an indicator of damage. While previous work had mostly focused on maximizing probability of detection, the present work demonstrated a more practical case where cost and risk were considered via a Bayes risk formulation, establishing a workflow for probabilistic reasoning. As shown, the optimal number of samples to be collected depends on the cost associated with each decision taken. While this work focused on a single example with multiple damage cases, the workflow developed is extensible and applicable to other structures, and other cost paradigms. Moreover, this work demonstrated how the model can be updated to new data, and how that updates reduce the predictive uncertainty significantly. Furthermore, the statistics of the updated weights were used to infer an understanding of the physical mechanism driving the change. Future work will focus on extensions to understand the physics of the damage mechanism with more detail and extensions to the cost function considered.

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## Chapter 8

# Model-Form Uncertainty in Predicting Electromechanical Passive and Active Damping Forces for a One-Mass Oscillator

Murali Krishnan Mohan Kumar and Roland Platz

**Abstract** This study investigates the passive and active damping behavior achieved by a voice coil actuator (VCA) for a one-mass oscillator subjected to passive and active vibration isolation. The novelty of this investigation lies in validating an adequate analytical model that predicts the passive and active damping properties provided by the VCA for the one-mass oscillator. In the existing analytical model, the VCA forces are functions solely of the passive damping coefficients as well as active gains and the velocities of the mass from a velocity feedback control to realize active damping. Initial verification and validation reveal discrepancies between the simulated numerical results and the experimental data. Experimentally, in higher damping scenarios, the resonance frequency of the one-mass oscillator shifts to higher values, whereas theoretically, it is expected to shift to lower values. The authors assume that the VCA's real behaviour is responsible for the shift. This motivates the following investigation. This study incorporates the electromechanical coupling of the VCA's inductance, resistance, and force constant with the mechanical damping and stiffness properties of the oscillator into the model.

**Keywords** Model-form uncertainty · Vibration isolation · Active and passive damping · Voice coil actuator · Electromechanical model

## Introduction

Model-form uncertainties occur when there is ambiguity in establishing a functional relationship between the input and output of the model. This ambiguity can stem from unknown, incomplete, inadequate, and unreasonable functional relationships between the inputs and the corresponding responses or model parameters and internal variables of the model. This model-form discrepancy results in incompleteness in defining the scope of application of the model, [1]. Hence, identification and quantification of uncertainty associated with a mathematical model helps in model selection and offers a comparison of their discrepancies. Such a comparison aids a designer in deciding the adequacy of a simple model over a complex model with unacceptable or acceptable uncertainty, [2].

An analytical one-mass oscillator model, as outlined in [2], was used as the basis for the initial investigations into uncertainty in passive and active vibration reduction. This study examined vibration isolation, distinguishing passive damping which relies on inherent system parameters such as inertia, stiffness, and damping and active damping, which uses additional controlled forces to enhance vibration isolation. Lenz and Platz [3] explored parametric uncertainty and validated the model with a test rig using an area validation matrix. Platz [2] identified the discrepancies between numerical one-mass oscillator and experimental results by the validation of the model for various damping cases. Further, a deterministic calibration of the one-mass oscillator along with a two-mass and a finite element (FE) model, representing the same experimental test set up was carried out in [4], keeping the damping ratio and stiffness as calibration parameters. [5] presented a mathematical model of a voice coil motor using the expression for LORENTZ FORCE incorporating the magnetic field density, current and geometric parameters of the coil. Banik et al. [6] optimised the design of a voice coil motor that can be used as an actuator for vibration isolation applications. The VCA model presented in [7], based on the Skyhook damper concept, was used for active vibration isolation using velocity feedback control. The electromechanical model of a one-mass oscillator in [8] accounts for the electromotive force of the actuator by considering its equivalent electric circuit parameters. Perfetto et al. [9] presented and validated a model of an active vibration absorber using a VCA with voltage driven feedback controller. This study investigated the actuator force, taking into account the electrical impedance of the VCA.

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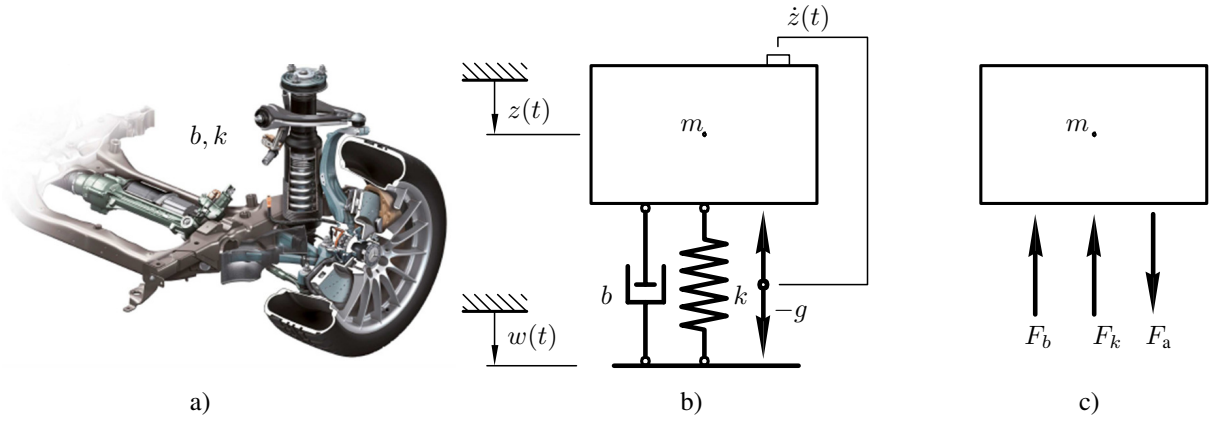
Murali Krishnan Mohan Kumar · Roland Platz

Department of Mechanical Engineering and Mechatronics, Deggendorf Institute of Technology, 91781 Weissenburg, Bavaria, Germany  
e-mail: [murali.mohan-kumar@th-deg.de](mailto:murali.mohan-kumar@th-deg.de); [roland.platz@th-deg.de](mailto:roland.platz@th-deg.de)

As an initial step to understand the discrepancies between the simulated results and experimental observations, the control voltage that creates the damping force generated by the VCA in the experimental test rig are examined in this paper. In the experiments, passive and active damping are realized through the control voltage of the VCA. However, the existing one-mass oscillator model treats the damping forces as pure mechanical forces, which are functions of the damping coefficient, active gain, and the velocities of the frame and mass. This work focuses on identifying discrepancies in the damping force predicted by the one-mass oscillator by incorporating electrical parameters, such as resistance, inductance, and the force constant of the VCA, based on the measured voltage values.

## Formulation of the Analytical One-Mass Oscillator Model and Realized Test Rig

The one-mass oscillator model is a single degree of freedom system as described in [2]. It serves as the foundation for investigations aimed at evaluating model-form uncertainty in passive and active vibration reduction. In an application scenario, the model represents a suspension leg as illustrated in Figure 1a.



**Fig. 1** Motivation for one-mass oscillator model from, a) front suspension leg (Mercedes Benz) with damping  $b$  and stiffness  $k$ , b) one-mass oscillator model with base excitation and active vibration isolation by active velocity feedback control, c) the passive and active forces acting on the oscillating mass,[3]

The suspension is modelled as a one-mass oscillator model in Figure 1b, where the passive configuration is characterized by the chassis mass  $m$  representing one quarter of a car, the stiffness  $k$ , and the damping coefficient  $b$  of the suspension leg, which is considered massless. In the active configuration, an additional velocity feedback gain  $g$  is introduced. The mass experiences an absolute vertical displacement  $z(t)$  in response to a base point displacement excitation function  $w(t)$ . The base point, assumed to have no inertia, is represented by a horizontal line in Figure 1b. According to the principle of linear momentum, from Figure 1c

$$m \ddot{z}(t) = F_a - F_b - F_k \quad (1)$$

where

$$F_a = -g \dot{z}(t), \quad F_b = b [\dot{z}(t) - \dot{w}(t)], \quad \text{and} \quad F_k = k [z(t) - w(t)] \quad (2)$$

are the active damping force, passive damping force, and stiffness force, respectively. The active force due to actuation arises from the velocity feedback, while the passive damping force depends on the passive damping coefficient  $b$  and the relative velocity  $\dot{z}(t) - \dot{w}(t)$  between the mass and the frame.

From (1) and (2), the equation for equilibrium of forces becomes

$$m \ddot{z}(t) + (b + g) \dot{z}(t) + k z(t) = b \dot{w}(t) + k w(t). \quad (3)$$

The inhomogeneous differential equation of motion

$$\ddot{z}(t) + \left[ 2 D_p \omega_0 + \frac{g}{m} \right] \dot{z}(t) + \omega_0^2 z(t) = 2 D_p \omega_0 \dot{w}(t) + \omega_0^2 w(t) \quad (4)$$

uses the passive damping ratio  $D_p$  and the angular eigen frequency  $\omega_0$  with

$$2D_p \omega_0 = \frac{b}{m} \text{ and } \omega_0^2 = \frac{k}{m}.$$

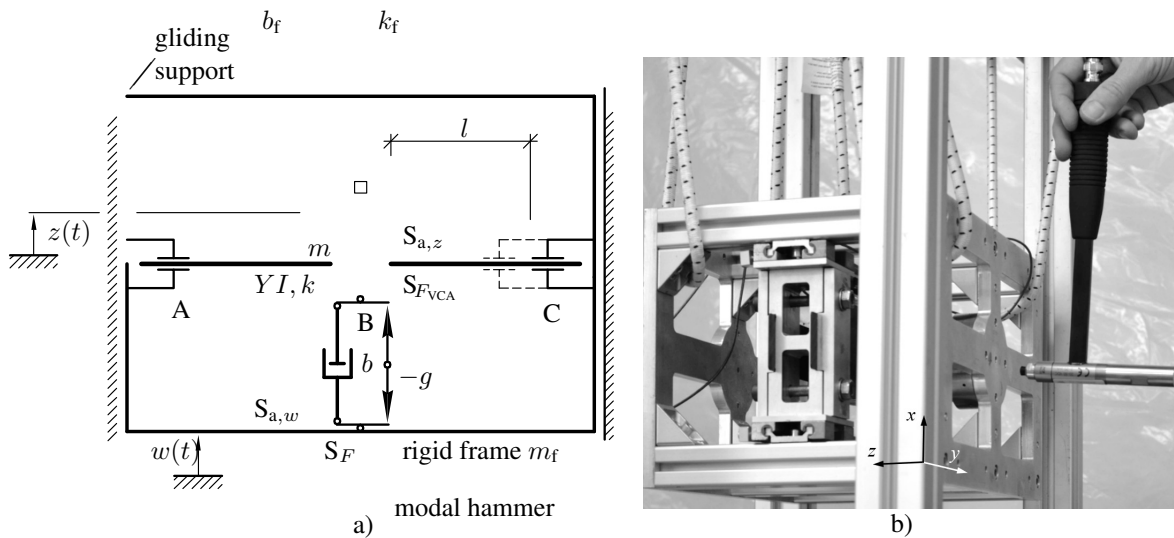
The passive and active damping forces resulting from (3) are  $2D_p \omega_0 m (\dot{z} - \dot{w})$  and  $g \dot{z}$ .

In the automotive application scenario illustrated in Figure 1a, the base point excitation  $w(t)$  in Figure 1b ideally represents driving on an uneven road. To simulate this condition in an experimental setup, an additional frame structure with mass  $m_f$ , damping coefficient  $b_f$ , and stiffness  $k_f$  was incorporated into the test rig, Figure 2, [2] and [4]. In the actual test setup, the single-mass oscillator is mounted within a frame of relatively large mass  $m_f$  compared to  $m$ . The model parameters used for the test rig are detailed in Table 1. The setup acts as the physical embodiment of the one-mass oscillator system shown in Figure 1b. A modal hammer applies the force on the frame, Figures 2a and 2b. It is then converted into a displacement excitation of the frame mass  $m_f$ . The test rig is supported by an elastic foundation with relatively low damping  $b_f$  and low stiffness  $k_f$ , Figure 2. The assumed damping coefficient  $b_f$ , with the corresponding damping ratio  $D_f = 0.1\%$ , and the stiffness  $k_f$  are relatively small compared to the damping ranges  $b$ , with the corresponding damping ratio  $D$  between 12 and 28 %, and the stiffness ranges  $k$  of the leaf springs, Table 1. This results in a quasi-static dynamic response of the frame following the hammer impact, characterized by a relatively low first eigenfrequency  $\omega_{0,f} \approx 2\pi \text{ 1/s}$ , which is much lower than the first eigenfrequency  $\omega_0$  of mass  $m$ .

**Table 1** Values of parameters used for the analyses, [4]

| Model parameters          | Variables  | Realized values            | Unit |
|---------------------------|------------|----------------------------|------|
| oscillating mass          | $m$        | 0.79 – 1.60                | kg   |
| damping coefficient       | $b$        | 39 – 93                    | Ns/m |
| stiffness                 | $k$        | 25,788 – 38,494            | N/m  |
| gain                      | $g$        | 15 – 84                    | Ns/m |
| frame mass                | $m_f$      | 10                         | kg   |
| frame damping coefficient | $b_f$      | 0.17                       | Ns/m |
| frame stiffness           | $k_f$      | 722                        | N/m  |
| eigenfrequency            | $\omega_0$ | $(26.2 - 36.7) \cdot 2\pi$ | 1/s  |

Figure 2a presents a schematic of the test rig. The rigid frame features two supports that secure a leaf spring at points A and C. The effective bending lengths on sides A–B and B–C are denoted as  $l$ , with the oscillating mass  $m$  positioned at



**Fig. 2** a) Concept of the experimental realization for vibration isolation – schematic diagram of the real test set up; b) realized test rig, [2]

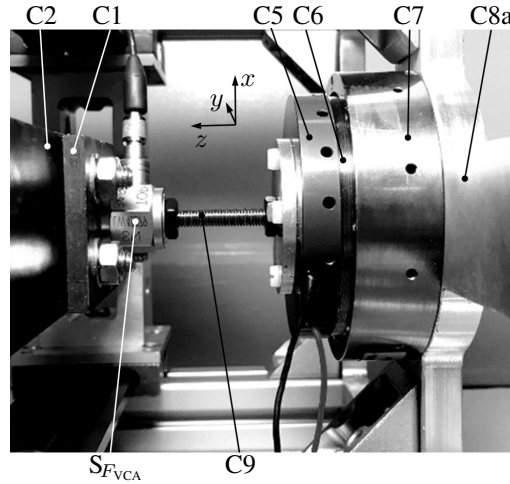


the center point B. The leaf spring has a cross-sectional area of  $d h$ , defined by its width  $d$  and height  $h$ . Its stiffness  $k$  is determined by the bending stiffness  $EI$ , where  $E$  represents the Young's modulus of the carbon fiber reinforced polymer (CFRP) used for the leaf spring, and  $I$  is the area moment of inertia. The two supports at A and C can be adjusted along  $l$  to fine-tune the leaf spring's bending deflection and, consequently, its effective stiffness  $k$ . A VCA generates the passive and active damping forces  $F_b$ ,  $F_a$  introduced in (2) and in Figure 1c. The force sensor  $S_{F_{VCA}}$  at B captures the sum of the forces  $F_a + F_b$  acting on the oscillating mass, Figure 2a.

Responses from the experimental tests are recorded using acceleration and force sensors  $S_a$  and  $S_F$ . Figure 2a illustrates the accelerometers  $S_{a,z}$  and  $S_{a,w}$ , which measure the accelerations of the oscillating mass  $m$  and the frame mass  $m_f$ , respectively. The force sensor  $S_{F_{VCA}}$  captures the damping force between  $m$  and  $m_f$ , while the force sensor  $S_F$  measures the force generated by the hammer impulse. The signals from these sensors are processed in the time domain using a real-time interface controller, along with high-pass/low-pass filters and a computer workstation, [2].

## Modelling of the Voice Coil Actuator

Figures 3, 4, and 5 illustrate various components of VCA on the experimental test rig. A VCA works on the principle of electromagnetism, where the interaction between an electric current in a coil and a magnetic field generates linear forces [8]. The VCA enables the passive damping and active gain forces  $F_b$  and  $F_a$  during the experiments, Figure 3. The VCA consists of two primary components. The first component is a stationary cylindrical permanent magnet with an outer ring, which is affixed to the frame as the north pole (N), accompanied by a centered pole piece that serves as the south pole (S), Figures 4 and 5. The second component is a coil wound around a hollow, movable cylindrical mount that fits into the gap between the north and south poles of the magnet. A small air gap of approximately 1 to 1.5 mm exists between the magnet's poles and the cylindrical mount containing the coil, Figure 4. This cylindrical mount with the coil can move freely along the fixed  $z$ -axis between the north and south poles of the magnet, [2]. An applied control voltage regulates the current flowing through the coils within the magnetic field, resulting in the motion of the cylindrical mount in  $z$ -direction.



**Fig. 3** VCA on the test rig: C1 rigid oscillating mass  $m$ , C2 one leaf spring with stiffness  $k/4$ , C3 fixed leaf spring support, C4 glide support, C5 VCA coil body, C6 VCA coil, C7 VCA stator, magnet outer ring, north pole, C8a front structure of rigid frame with mass  $m_f$ , C9 rigid coupling screw,  $S_{F_{VCA}}$  force sensor to measure the sum of damping forces  $F_b$  and  $F_a$

The magnet in Figure 4 provides a magnetic flux density  $\vec{B}$ , an electrical current  $\vec{i}$  runs through the coil when applying a control voltage  $u$ . Both  $\vec{B}$  and  $\vec{i}$  are circumferential vectors around the  $z$ -axis with magnitude and direction. With the given direction of the current  $\vec{i}$  in the plan view A-A in Figure 4, the LORENTZ force vector, [5] and [10],

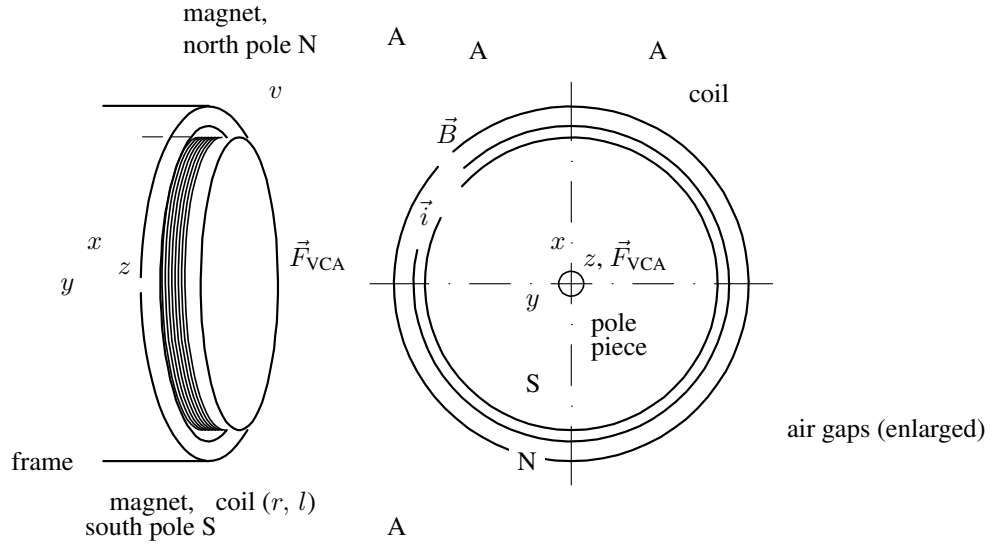
$$\vec{F}_{VCA} = r l \vec{i} \times \vec{B} \quad (5)$$

is given by the cross product of the current vector  $\vec{i}$ , multiplied by the length of the coil  $l$  and the ratio  $r$  of the effective length, and the magnetic flux density vector  $\vec{B}$ . Since  $\vec{i} \perp \vec{B}$ , the LORENTZ force becomes a scalar, [6],

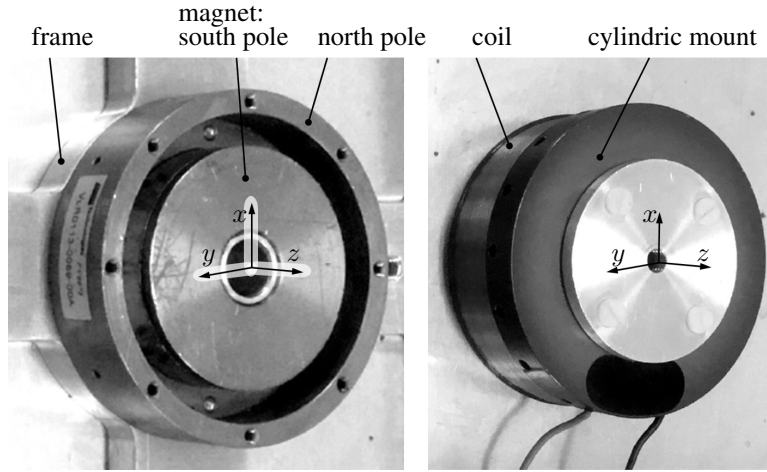
$$F_{VCA} = r l i \cdot B = \Psi i \quad (6)$$

with the force constant  $\Psi$ . In this work, this LORENTZ force is the electromotive force which accounts for the passive and active gain forces in (2), [2]. Hence,

$$F_{VCA} = b(\dot{z} - \dot{w}) + g\dot{z} = \Psi i. \quad (7)$$



**Fig. 4** Principal assembly of the VCA



**Fig. 5** Photo of VCA used in the rig

The driving electrical power of the VCA is

$$P = u_{VCA} i = F_{VCA} v \quad (8)$$

obtained by multiplying the current  $i$  with the driving voltage  $u_{VCA}$ . This is equivalent to the driving mechanical power, which can be expressed as the product of the electromotive force  $F_{VCA}$  and the velocity  $v$  of the coil's movement in the  $z$ -direction, Figure 4.

The electric equivalent circuit diagram of the VCA in Figure 6, [2] and [8], helps to understand the relation between the electrical and the mechanical quantities to, eventually, determine the electromotive force  $F_{VCA}$  (7) from an applied control voltage  $u$ . The VCA's properties are the inductance  $L$ , the Ohmic resistance  $R$ , and the force constant  $\Psi$ . Using the mesh rule, the applied control voltage becomes

$$u = u_{VCA} + u_L + u_R. \quad (9)$$



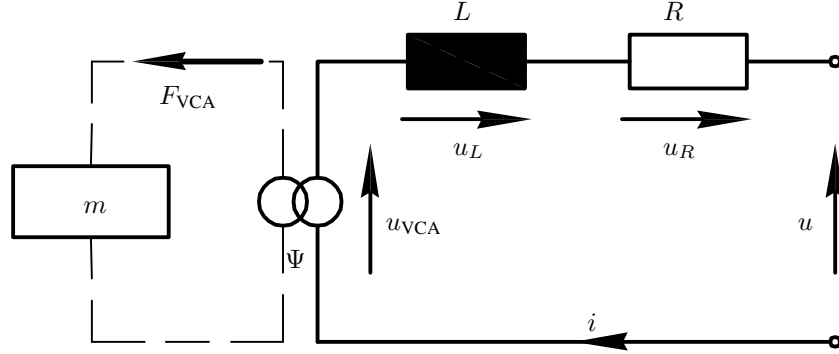
The voltages at  $L$  and  $R$  are

$$u_L = \frac{di}{dt}L \text{ and } u_R = iR, \quad (10)$$

and at  $\Psi$ , the driving voltage

$$u_{VCA} = \Psi v \quad (11)$$

initiates the electromotive force  $F_{VCA}$ .



**Fig. 6** Equivalent circuit of VCA to provide the electromotive force  $F_{VCA}$  as damping and active forces on the vibrating mass  $m$

Hence, the first order differential equation determines the applied control voltage

$$u = \Psi v + \frac{di}{dt}L + iR \quad (12)$$

as the sum of all voltages in the circuit in Figure 6. From (7), current

$$i = \frac{1}{\Psi}(b(\dot{z} - \dot{w}) + g\dot{z}) \quad (13)$$

is expressed in terms of force constant  $\Psi$ , the relative velocity between the magnet, which is fixed to the frame, and the coil winding, which is fixed to the moving mass  $m$ . The control voltage becomes, [2],

$$u = \left\{ \Psi(\dot{z} - \dot{w}) + \frac{d}{dt} \left[ \frac{1}{\Psi}(b(\dot{z} - \dot{w}) + g\dot{z}) \right] L + \frac{1}{\Psi}(b(\dot{z} - \dot{w}) + g\dot{z})R \right\}. \quad (14)$$

This equation is analogous to the equation for voltage generated by feedback control, as suggested by [9], where the control voltage is represented as the sum of three contributions, each given by a gain multiplied by velocity. The first term,  $\Psi(\dot{z} - \dot{w})$ , represents the compensator, which is necessary to eliminate the effects of a back electromotive force between the coil and the magnet, [9].

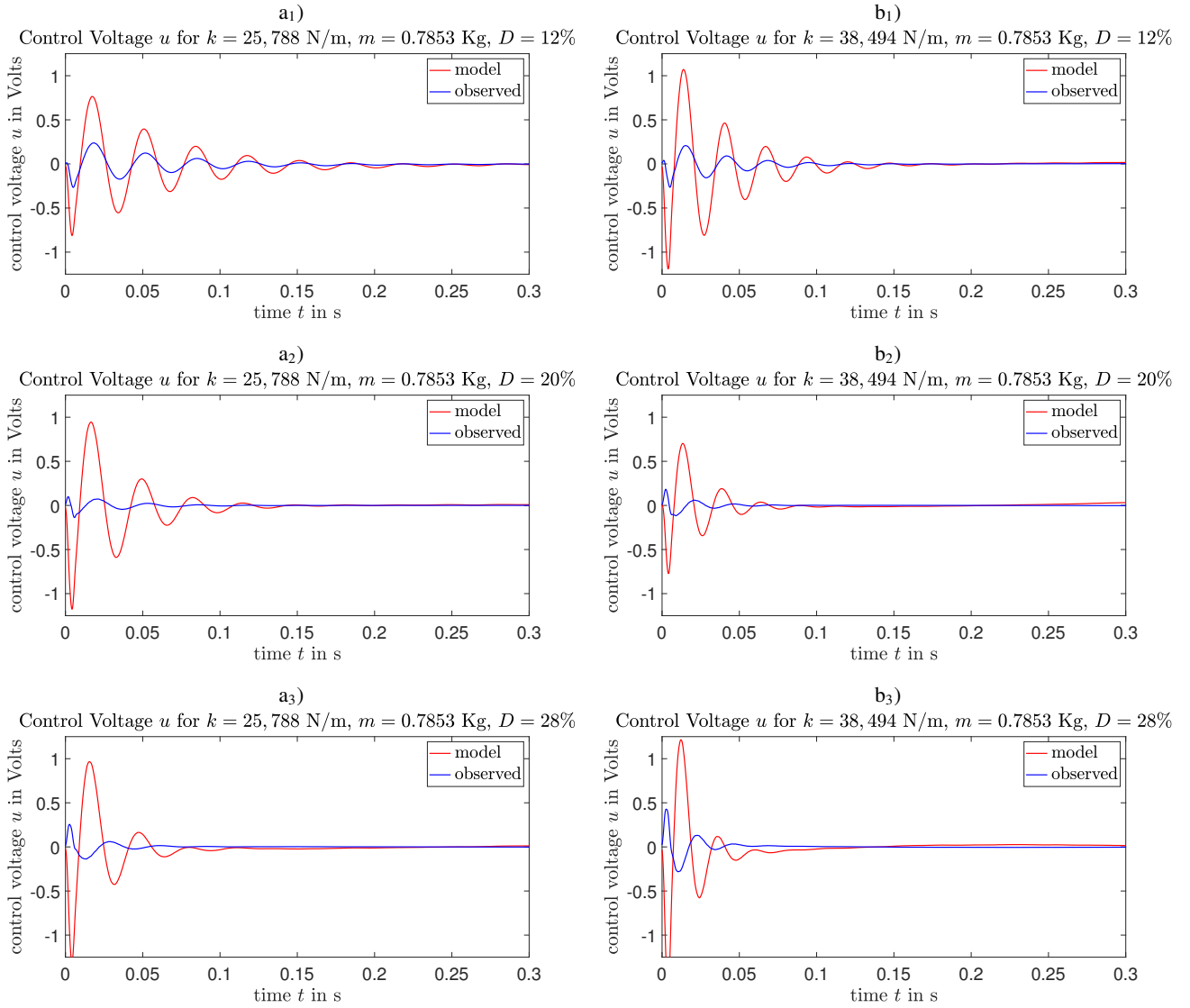
## Validation of Control Voltage Using Simulated and Experimental Comparisons

To validate the control voltage (14), calibrated values of the damping coefficient  $b$ , active gain  $g$ , and stiffness  $k$  from the one-mass oscillator in Figure 1 according to [4] are used. The simulated voltages from the model are compared with the measured control voltage obtained from experimental tests conducted for combinations of the mass  $m = 0.7853$  kg and three damping cases  $D = 12\%$ ,  $20\%$  and  $28\%$ , Figure 7, for stiffness values

- a)  $k = 25,788$  N/m and
- b)  $k = 38,494$  N/m.

The gain is set to zero,  $g = 0$ , only passive cases are considered.

The comparison between the simulated and experimental control voltages in Figure 7 reveals deviations, particularly in amplitude across all cases. The simulated voltage appears to be a scaled version of the actual voltage, especially for the lowest and highest passive damping ratios of  $12\%$  and  $28\%$ . In the case of the  $20\%$  damping ratio, the results show deviations in both amplitude and phase. It can also be observed that the experimental values for the  $20\%$  and  $28\%$  damping ratios are in



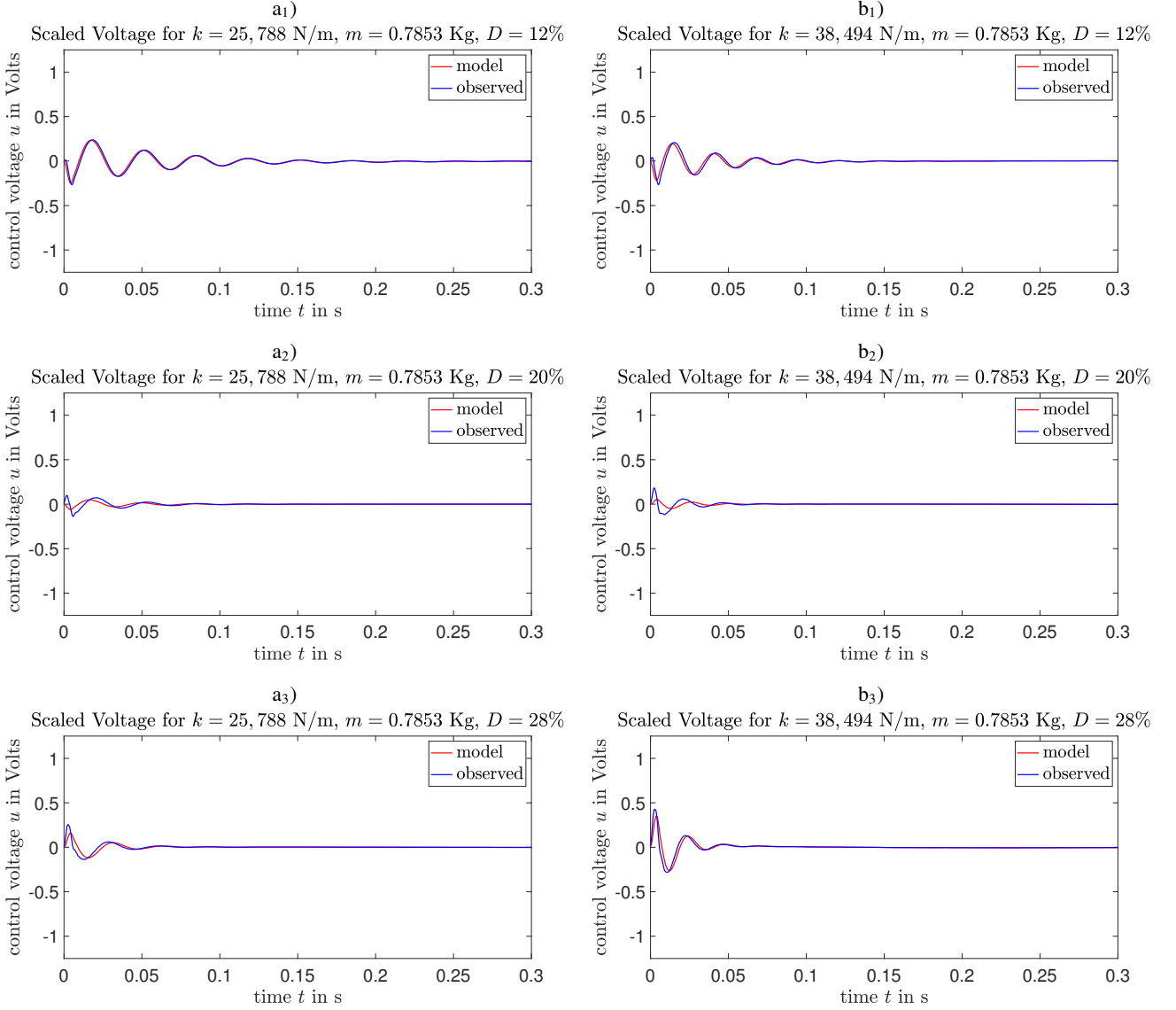
**Fig. 7** Comparison of the observed control voltage and the model predictions for a)  $k = 25,788$  N/m, and b)  $k = 38,494$  N/m for the mass  $m = 0.7853$  Kg and passive damping ratios  $D = 12\%$ ,  $20\%$ ,  $28\%$

**Table 2** Suggested scale of simulated control voltage amplitudes in comparison to observed values

| Damping Cases | $k = 25,788$ N/m, $m = 0.7853$ kg | $k = 38,494$ N/m, $m = 0.7853$ kg |
|---------------|-----------------------------------|-----------------------------------|
|               | Scale                             | Scale                             |
| $D_1 = 12\%$  | 2.55                              | 8.07                              |
| $D_2 = 20\%$  | -6.87                             | -15.00                            |
| $D_3 = 28\%$  | -8.43                             | -4.45                             |

reverse phase with the curve for the 12% damping ratio. However, the simulated curve for all three damping cases initially proceeds into the negative region and follows a similar trend.

A scaling factor for the amplitude is suggested in each case to match the model with the observations, Table 2. The values do not show a consistent trend across the passive damping ratios for the two test cases, though. Figure 8 shows the scaled plot of the simulated control voltage in comparison to the experimentally observed values. The scaled plot almost fits for the lowest and the highest damping values in both test cases. For the intermediate damping of 20%, adequate fit is not possible.



**Fig. 8** Comparison of the observed control voltage and the scaled model predictions for a)  $k = 25,788$  N/m, and b)  $k = 38,494$  N/m for the mass  $m = 0.7853$  Kg and passive damping ratios  $D = 12\%$ ,  $20\%$ ,  $28\%$

## Conclusion

To reduce and identify model-form uncertainty in predicting the dynamic response of an analytical one-mass oscillator model, a VCA providing passive and active vibration isolation in the test rig was thoroughly investigated. An electromechanical model of the VCA was adapted, capturing the key mechanical and electrical components responsible for generating the damping force. The control voltage was expressed as a function of the damping ratio, stiffness, velocities of the frame and mass, along with the electronic parameters such as inductance, resistance, and the actuator's force constant. A comparison between the simulated and experimentally observed voltages for the passive cases revealed discrepancies, which were analyzed by determining the appropriate scale factor. This attempt aims to improve the accuracy of the system's dynamic response and lower the model-form uncertainty. However, the scaling factor is purely empiric by observation, without any physical explanation yet. Future work will investigate the discrepancy between the experimental and the modelled outcome of the control voltage to find a rather physical explanation.

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## Chapter 9

# Influence of Sensor Uncertainty on Frequency Response Functions of Hydraulic Accumulators

Oscar L. Lefemmine, Manuel Rexer, Maximilian M. G. Kuhr, and Peter F. Pelz

**Abstract** The determination of the frequency response function (FRF) of hydraulic components, such as hydraulic accumulators, is typically carried out in the frequency domain. However, due to stochastic and systematic measurement errors, the obtained FRF, i.e., the accumulator's frequency-dependent stiffness, is also subject to uncertainty. In particular, the systematic measurement uncertainty of the sensors used, which is typically neglected during uncertainty propagation into the frequency domain, is crucial for determining the uncertainty value of the frequency-dependent stiffness. At the Chair of Fluid Systems, investigations are focused on optimizing the energy density of hydraulic accumulators. Within the scope of this paper, PUR foam is introduced into the gas-side of the accumulator, increasing its thermal capacity. The foam's porous structure, provides a large area for heat transfer, leading to a reduction in stiffness while maintaining the same accumulator geometry. To predict the dynamic behavior, analytical models are set up and compared to experimental measurement data gathered with an one-axis servo-hydraulic test rig, focusing on the pressure response to harmonic volume displacement. In order to validate the analytical models, it is indispensable to determine the uncertainty in measurement data. Therefore, this paper focuses on applying an uncertainty quantification framework, presented previously at IMAC-XLII, to quantify the impact of different types of uncertainty on the measurement results. The framework uses different sensor error models in combination with Monte Carlo simulations to propagate the modeled errors into the frequency domain. This approach is applied to evaluate measurements taken on hydraulic accumulators with PUR foam infill, investigating the influence of uncertainty.

**Keywords** Uncertainty · Systematic Error · Frequency Domain · Monte Carlo Simulation · Hydraulic accumulators

## Introduction

The behavior of dynamic systems is determined by their intrinsic structure, which can be described using spring, damper and mass models and characterized by parameters such as stiffness and damping coefficients. Dynamic systems are found in various forms across many fields. From bridges to powertrains and the periodic detachment of cavitation clouds, understanding the response to external excitation is crucial for the design and operation of these systems. The determination of dynamic behavior is achieved through analytical models, simulations, or experiments. Since experimental investigations are often considered preferable [1, 2], knowledge of the propagation of uncertainty inherently linked to the use of measurement equipment is essential for accurate interpretation of results. A thorough understanding of a dynamic system's behavior enables optimal design and control [3].

Although determining dynamic characteristics in the frequency domain is standard practice, proper quantification of the uncertainty in the identified quantities is often lacking. This is primarily due to the complex propagation of both random and systematic uncertainty from the time domain to the frequency domain, or vice versa. Generally, the Joint Committee for Guides in Metrology (JCGM) [4] distinguishes between two types of uncertainty: type A uncertainty, i.e., normally distributed stochastic uncertainty, and type B uncertainty, i.e., systematic measurement uncertainty with unknown distribution. Extensive literature exists on the propagation of type A uncertainty into the frequency domain [5–9]. In contrast, for type B uncertainty, only conservative assumptions are typically made, as investigated by Kuhr [10]. According to JCGM, type A uncertainty is handled using Gaussian uncertainty propagation, whereas type B uncertainty require more sophisticated approaches, such as Monte Carlo simulations [4].

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Oscar L. Lefemmine · Manuel Rexer · Maximilian M. G. Kuhr · Peter F. Pelz  
Chair of Fluid Systems, Technische Universität Darmstadt, 64287 Darmstadt, Germany  
e-mail: [oscar.lefemmine@tu-darmstadt.de](mailto:oscar.lefemmine@tu-darmstadt.de); [manuel.rexer@tu-darmstadt.de](mailto:manuel.rexer@tu-darmstadt.de); [maximilian.kuhr@tu-darmstadt.de](mailto:maximilian.kuhr@tu-darmstadt.de);  
[peter.pelz@tu-darmstadt.de](mailto:peter.pelz@tu-darmstadt.de)

Rexer et al. [11] used Monte Carlo simulations in their investigations to determine simple propagation laws of sensor uncertainty into the frequency domain and, thus, the influence on the magnitude and phase of complex Fourier coefficients. In this paper, the influence of systematic sensor errors on the frequency-dependent stiffness of hydraulic accumulators is investigated by applying the propagation laws found by Rexer et al. [11]. First, some background information on the propagation modeling will be provided. In addition to that, some basics on the frequency-dependent stiffness of gas volumes will be described, with particular focus on the physical effects occurring at different frequencies. To present the experimental procedure, the test rig and the experimental setup are explained in more detail. In conclusion, the results of the error propagation will be presented and discussed.

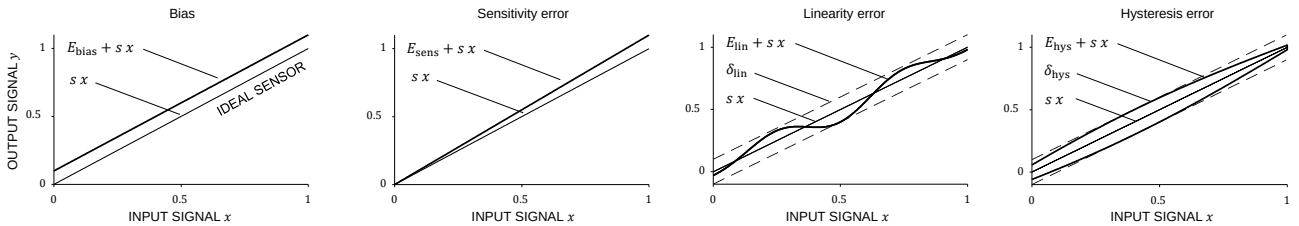
## Uncertainty propagation into frequency domain

To characterize the response of hydraulic accumulators to harmonic excitation, pressure and temperature measurements within the accumulator are used to determine its frequency-dependent stiffness. Following the approach outlined by Puff [12] and Hedrich et al. [13, 14], these signals are transformed into the frequency domain using Fast Fourier Transforms (FFT), allowing the evaluation of the accumulator's stiffness across different excitation frequencies.

The following description of error propagation is based on the methods of Rexer et al. [11]. The use of sensors is directly associated with inaccuracies in the recorded measurements. The transfer behavior cannot be considered ideal, so a sensor error  $E_i = y_i - x_i$  is to be expected between the measured value  $y_i$  and the actual value  $x_i$ . While this kind of error represents a specific deviation, unknown measurement errors can be treated statistically using probability distributions. This approach enables the quantification of uncertainty, offering a framework for analysis through statistical methods. In other words, the uncertainty  $\delta_i$  of the measured value  $y_i$  defines an interval around the mean value  $\bar{y}$  within which the true value is likely to fall. Hence,  $y_i = \bar{y}_i \pm \delta_{95}(y_i)$  with a confidence interval for  $\delta_i$  of 95 %.

Generally, sensor errors can be divided into random errors and systematic errors, with the latter further classified into bias or offset, sensitivity, linearity, and hysteresis errors [15], cf. fig. 1. Data sheets provided by manufacturers often address errors by providing deviation bounds,  $\delta_{lin}$  and  $\delta_{hys}$ , as shown in fig. 1, within which the measured value is expected to fall. However, the distribution is unknown and can therefore be described as incertitude according to Pelz et al. [3]. To utilize Monte Carlo simulations, generalized sensor and error models are required. Typically, sensor transfer behavior is approximated using an ideal characteristic curve  $y$  as a function of the input signal  $x$ , with slope  $s$  and offset  $o$ . To account for the four types of sensor errors, additional terms  $fn(x, \dot{x})$  are introduced. It is assumed that linearity errors vary with the input signal, thus  $E_{lin} = fn(x)$ , and that hysteresis errors depend on both the input signal and the direction from which the value is approached, hence  $E_{hys} = fn(x, \dot{x})$ . The sensor model can then be described with

$$y = sx + o + E_{bias} + E_s x + E_{lin}(x) + E_{hys}(x, \dot{x}). \quad (1)$$



**Fig. 1** Different types of sensor errors described in [15]. Offset errors result in a static deviation in the output signal and should be differentiated from sensor offsets defined by their measuring range. Sensitivity errors show a deviation from the sensor's ideal slope. Linearity errors and hysteresis errors can take various, complex forms [11].

Bias and sensitivity errors can be easily modeled using constant values  $E_{bias}$  and  $E_s$ . In the latter case,  $E_s$  represents the slope of the faulty characteristic curve. The values of these constants, without the corresponding probability function, are typically provided in the sensor's data sheet. To propagate uncertainty effectively rather than just performing interval estimation, the JCGM suggests assuming a uniform distribution as a conservative estimate [4]. Both uncertainties can be



directly propagated into the frequency domain

$$\tilde{\delta}_{\text{bias}}(|Y(\omega)|) = \begin{cases} \delta_{\text{bias}}, & \omega = 0 \\ 0, \forall & \omega \neq 0 \end{cases}, \quad (2)$$

$$\tilde{\delta}_s(|Y(\omega)|) = \delta_s. \quad (3)$$

Regarding linearity and hysteresis errors, modeling their behavior is more complex. To describe the influence on a measured signal, Rexer et al. [11] follow an approach based on Fourier transformation, where an arbitrary signal can be broken down into a series of harmonic signals. Hence, to describe linearity errors in eq. (4), a sum of  $M$  sinusoidal signals is used

$$E_{\text{lin}}(x) = \sum_i a_i \sin(w_i x + \varphi_i), \quad i = 1 \dots M \in \mathbb{N}_0. \quad (4)$$

In this case, the parameters  $a_i$ ,  $w_i$ , and  $\varphi_i$ , i.e., amplitude, frequency, and phase shift respectively, are considered statistically distributed quantities, cf. table 1. In order to restrict values calculated with eq. (4) within the given sensor bounds stated in the manufacture's data sheet, the equation is scaled with

$$\max[E_{\text{lin}}(x)] \leq \delta_{\text{lin}}. \quad (5)$$

Hysteresis errors are modeled in a similar way. In the case of sensors, hysteresis describes the effect of different measured values depending on the direction of approach. This is taken into account by including the change of the measured signal over time,  $\dot{x}$ , and approximating the signum function with the hyperbolic tangent function. Analogous to linearity errors, a sum of harmonic signals is used to describe hysteresis errors

$$E_{\text{hys}}(x, \dot{x}) = \sum_i \tanh(\dot{x}) a_i \sin(w_i x + \varphi_i), \quad i = 1, \dots, M \in \mathbb{N}_0. \quad (6)$$

**Table 1** Values used to parametrize the error model given by eq. (4) and eq. (6), as well as the test signal, eq. (8), for the Monte Carlo simulation.

|                           | Parameter           | Value            | Distribution |
|---------------------------|---------------------|------------------|--------------|
| Amplitude (error)         | $a_i$               | $U[0, 1]$        | uniform      |
| Phase shift (error)       | $\varphi_i$         | $U[0, 2\pi]$     | uniform      |
| Wavelength                | $w_i$               | $ N(0, \sigma) $ | half-normal  |
| Sample frequency          | $f_{\text{sample}}$ | 1 kHz            | -            |
| Duration                  | $L$                 | 1 s              | -            |
| Signal frequency          | $\Omega$            | 10 Hz            | -            |
| Offset                    | $o$                 | 10               | -            |
| Amplitude (test signal)   | $\hat{a}$           | 1                | -            |
| Phase shift (test signal) | $\Phi$              | $45^\circ$       | -            |

For preliminary investigations, the distribution of the parameters  $a_i$ ,  $w_i$ , and  $\varphi_i$  was set up according to table 1. Different Monte Carlo simulations for the linearity error  $E_{\text{lin}}$  showed that a uniform distribution could not be realized by changing the parameters  $a_i$ ,  $w_i$ , and  $\varphi_i$ , or the values of their respective distributions. fig. 2 illustrates how the distribution changes, for example, from an arcsine distribution to a normal distribution when the standard deviation of the wavelength  $w_i$  is varied. Hence, the assumptions according to JCGM cannot be met. To generate a uniform distribution of the linearity error  $E_{\text{lin}}$ , Rexer et al. [11] adopt the approach of describing the amplitude  $a_i$  as a function of the wavelength  $w_i$  in the form of

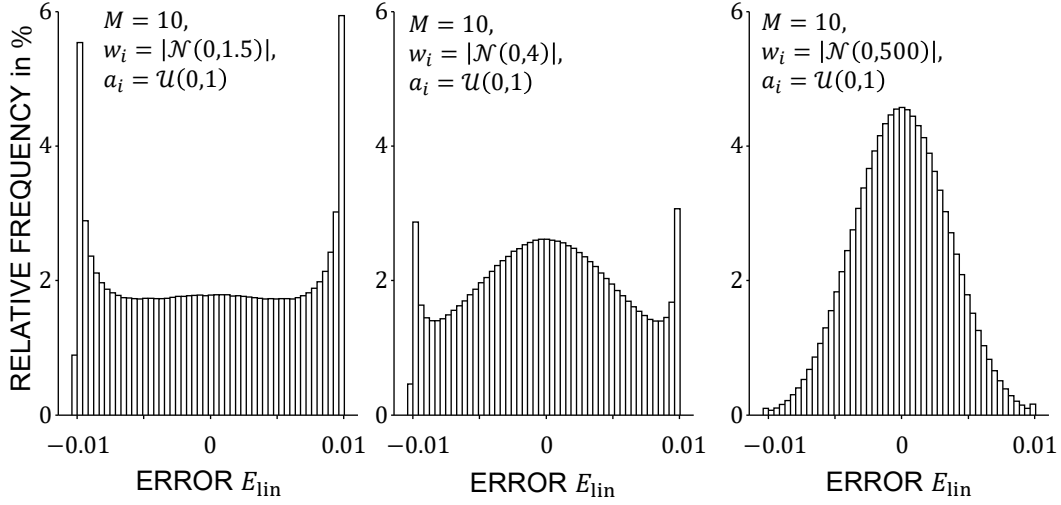
$$a_i = \frac{1}{w_i^2}. \quad (7)$$

Performing Monte Carlo simulations using the modification in eq. (7) led to a uniform distribution of the linearity error, cf. fig. 3.

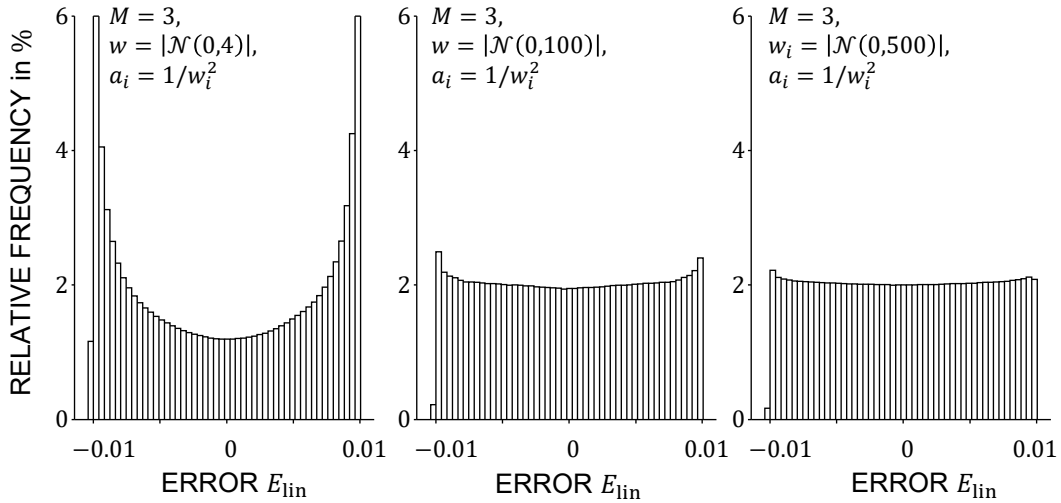
To investigate the influence of sensor errors on the frequency response, a test signal must be generated. Hence, a generic harmonic input signal is used to feed the sensor model

$$x = o + \hat{a} \sin(2\pi\Omega t + \phi). \quad (8)$$

In eq. (8),  $o$  is an offset,  $\hat{a}$  the signal amplitude,  $\Omega$  the excitation frequency, and  $\phi$  a phase shift. Both models, i.e., eq. (1) and eq. (8), are used in a Monte Carlo simulation to investigate the influence on the magnitude and phase of the frequency response. For this, the test signal is characterized according to table 1.



**Fig. 2** Distribution of linearity error resulting from the Monte Carlo simulation using parameters in table 1 [11]. The figure shows that the distribution changes from an arcsine distribution to a normal distribution as the standard deviation of the uniform distribution of the wavelength  $w_i$  increases. The arcsine distribution results from a harmonic sinusoidal function being sampled with equidistant time steps.



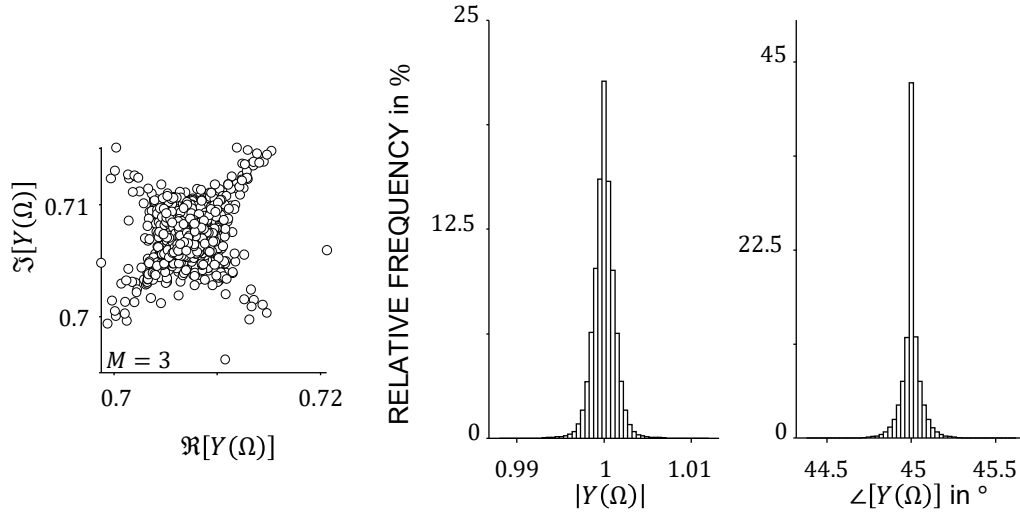
**Fig. 3** Resulting distribution using parameters in table 1 and eq. (7) [11]. The distribution changes from an arcsine distribution to a uniform shape. Hence, the JCGM's suggestion of assuming a uniform distribution as a conservative estimation is met.

After having modeled the sensor characteristic and the test signal, a final Monte Carlo simulation demonstrates the influence on the magnitude and phase of the frequency response function. To achieve this, errors are generated randomly, and the resulting output signal  $y$  is then Fourier-transformed. A total of  $10^6$  iterations are performed to obtain the results shown in fig. 4. The cross-shaped form of the scatter plot indicates that the linearity and hysteresis errors are independent of each other in the frequency domain. A sensitivity analysis varying the sensor error bounds  $\delta_{\text{lin}}$  and  $\delta_{\text{hys}}$  reveals the linear dependency of the systematic error for both magnitude and phase on  $\delta_{\text{lin}}$  and  $\delta_{\text{hys}}$ , respectively. From fig. 5, two propagation

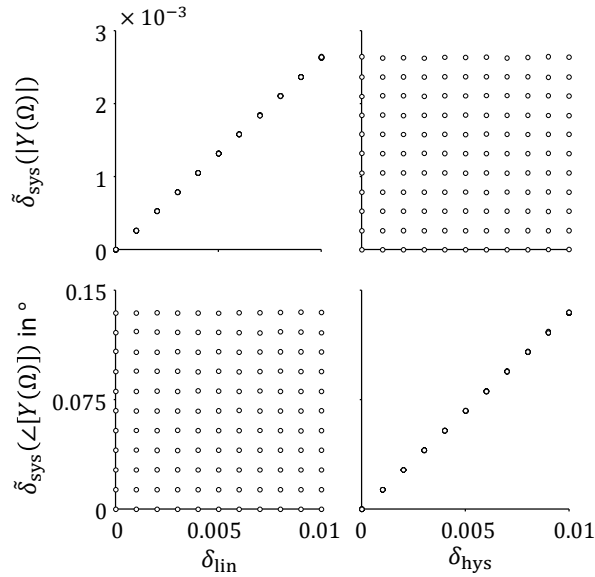
laws can be deduced:

$$\tilde{\delta}_{\text{lin}}(|Y(\Omega)|) = 0.263 \delta_{\text{lin}}(y), \quad (9)$$

$$\tilde{\delta}_{\text{hys}}(\angle[Y(\Omega)]) = \tan^{-1} \left( \frac{0.235 \delta_{\text{hys}}(y)}{|Y(\Omega)|} \right). \quad (10)$$



**Fig. 4** Results of the Monte Carlo simulation with the final determined parameters [11].

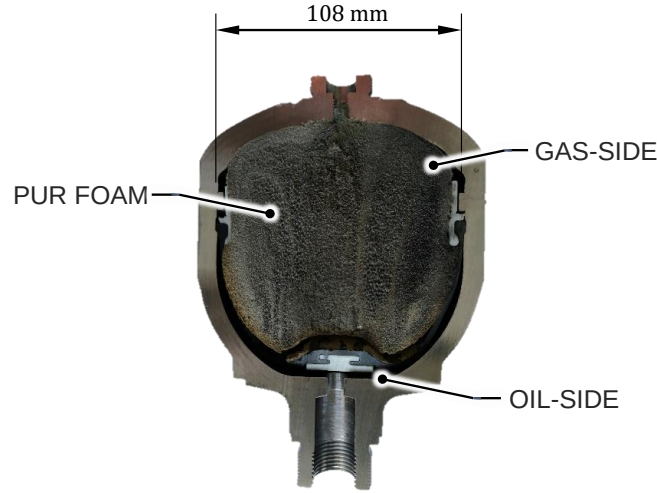


**Fig. 5** Influence of the uncertainty intervals of the linearity and hysteresis errors on the standard deviation of magnitude and phase of the Fourier coefficient  $Y(\Omega)$  [11].

After modeling sensor errors such as bias, sensitivity, linearity, and hysteresis, and simulating their propagation into the frequency domain using Monte Carlo simulations, the influence on the measurement process can be quantified. The methodology demonstrated how systematic errors affect both the magnitude and phase of frequency response functions and showed a linear dependency on their respective uncertainty bounds. With this groundwork laid, basic knowledge about the frequency-dependent stiffness of hydraulic accumulators is given.

## Frequency-dependent stiffness of hydraulic accumulators

Hydraulic accumulators function as conservative spring elements within hydraulic systems. Commercial vehicles like tractors, wheel loaders, and dump trucks often utilize compressed gas for suspension. Since hydraulic oil systems are commonly integrated into these vehicles, hydro-pneumatic suspensions can be implemented. In spring-mass systems, lower spring stiffness results in better vibration isolation. However, stiffness decreases as the volume of the hydraulic accumulator increases [16, 17]. Components for vehicle suspensions must be small and light due to space constraints and the need to minimize weight, which affects energy consumption for propulsion. The integration of porous materials offers an effective approach to increasing the thermal capacity of hydraulic accumulators, enabling a reduction in size without compromising stiffness. In diaphragm accumulators, pressure is applied to the oil side of a flexible diaphragm, compressing the gas, typically nitrogen, on the other side. Figure 6 illustrates the interior of an accumulator with foam infill.



**Fig. 6** Cross-sectional view of a hydraulic accumulator with PUR foam infill.

Accumulators are typically prefilled with a gas to a specified precharge pressure  $p_0$ , occupying an initial volume  $V_0$ . They are subsequently pressurized with hydraulic fluid to the operating pressure  $p_1$  and then subjected to oscillatory volume changes at an angular frequency  $\Omega$ , leading to time-dependent pressure  $p$  and volume  $V$ . These parameters form the basis for describing the dynamic stiffness of the gas volume. The dynamic behavior of gas volumes changes depending on the excitation frequency, resulting in increased stiffness as the frequency rises. Pelz and Buttenbender [17] provide a foundational model for the frequency-dependent stiffness. While their model does not account for the influence of porous materials, studies by Hartig [16], Rexer [18], Rexer et al. [19], and Pelz et al. [20] provide insights into the interactions between gas and foams, activated carbon, and other porous materials in both air springs and hydraulic accumulators. An increase in stiffness is observed as the state change transitions from isothermal at low excitation frequencies to adiabatic at high frequencies. Hartig [16] and Rexer [18] identify different plateaus in their stiffness models, which are attributed to changing interactions between gas and infill material.

The frequency-dependent stiffness  $K$  is defined as the change in pressure  $dp$  with respect to a periodically applied volumetric displacement  $dV$ . In order to enable a meaningful comparison between different types of hydraulic accumulators, it is useful to non-dimensionalize the stiffness  $K$ . For this purpose, the pressure  $p_1$  and the corresponding volume  $V_1$  of the accumulator are used. Introducing the dimensionless variables  $p = p^+/p_1$  and  $V = V^+/V_1$ , the dimensionless stiffness is defined as:

$$K^+ = \frac{p^+}{V^+}. \quad (11)$$

The frequency is non-dimensionalized in a similar manner, using a characteristic length  $L$ , thermal conductivity  $\lambda$ , gas density  $\rho$ , and specific heat capacity at constant pressure  $c_p$ . The resulting dimensionless frequency corresponds to the Péclet number:

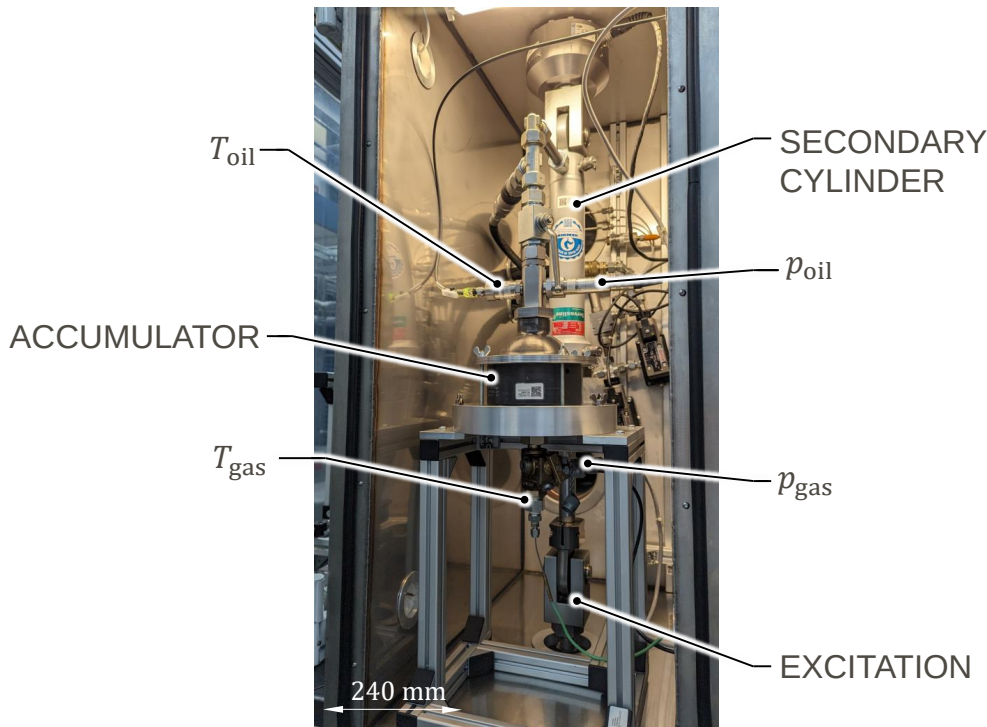
$$Pe = \frac{\Omega L^2}{\lambda/(\rho c_p)}. \quad (12)$$

## Experimental setup

For the experimental evaluation of the frequency-dependent stiffness, a HYDAC hydraulic accumulator (SBO450E-0.6A6-112U-450AK) with a nominal volume of 600 ml is utilized. Prior to integrating the accumulator into the test rig, pieces of PUR foam with two different porosities, measured in pores per inch (PPI), were shaped to fit the accumulator's contours and then inserted into the gas-side chamber. Nitrogen was then charged into the accumulator up to 40 bar. Before starting the harmonic excitation, an operating pressure of 60 bar was applied to the oil-side chamber via an external hydraulic system. The test rig features a temperature-controlled measurement chamber, enabling the setting of the ambient temperature, cf. fig. 7. The experiment started once thermal equilibrium was achieved, with the gas temperature matching the set ambient temperature. A sinusoidal excitation with parameters given in table 2 was applied to the accumulator, and the pressure response was measured at various frequencies. The measurement data is available on the European database Zenodo [21–24].

**Table 2** Parameters used for the excitation signal of the test rig. The cylinder of the one-axis servo-hydraulic test rig performs a harmonic sinusoidal motion, resulting in a volumetric displacement of the oil in the accumulator. Hydraulic accumulators are typically prefilled with nitrogen and pressurized to the operating pressure.

|                    | Value       |
|--------------------|-------------|
| Amplitude          | 2 mm        |
| Frequency          | 0.1 – 20 Hz |
| Prepressure $N_2$  | 40 bar      |
| Operating Pressure | 60 bar      |



**Fig. 7** Image of the hydraulic accumulator under investigation, installed in the test rig. The cylinder of the one-axis servo-hydraulic test rig moves a secondary cylinder that causes the displacement of a defined volume of oil. The entire assembly is situated in a temperature-controlled chamber that is connected to an external climatic chamber.

## Results and discussion

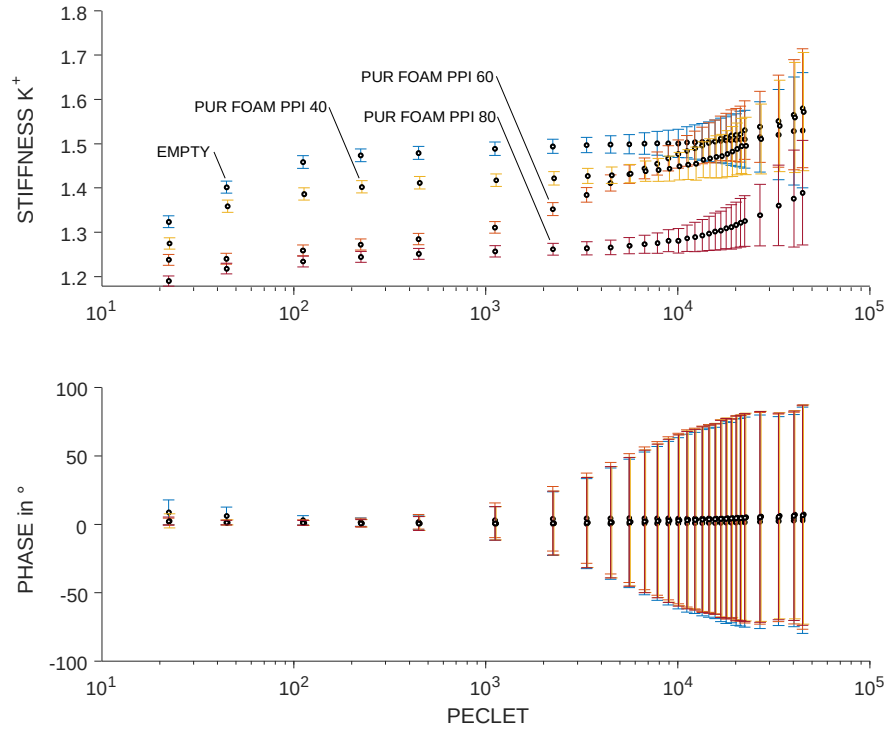
For the evaluation of the accumulator's stiffness, a MATLAB based software framework for calculating uncertainty in the frequency domain is used [25]. The measured sensor signals for pressure and cylinder displacement are subject to sensor errors. Consequently, calculated values, such as the dimensionless stiffness, are also subject to uncertainty, where these uncertainties propagate into the frequency domain. The uncertainties considered are calculated according to eq. (13) and eq. (14).

$$\tilde{\delta}_{\text{sys}}(|Y(\Omega)|) = \begin{cases} \tilde{\delta}_{\text{bias}} + \tilde{\delta}_{\text{sens}} + \tilde{\delta}_{\text{lin}}, & \Omega = 0 \\ \tilde{\delta}_{\text{sens}} + \tilde{\delta}_{\text{lin}}, & \forall \Omega \neq 0 \end{cases}, \quad (13)$$

$$\tilde{\delta}_{\text{sys}}(\angle[Y(\Omega)]) = \begin{cases} 0, & \Omega = 0 \\ \tilde{\delta}_{\text{hys}}, & \forall \Omega \neq 0 \end{cases}. \quad (14)$$

The results in fig. 8 show the dimensionless stiffness of an empty hydraulic accumulator with a nominal capacity of 600 ml, as well as with three types of porous materials. The upper blue data points correspond to an accumulator without infill. It can be seen that introducing a porous material into the accumulator significantly changes its properties. With increasing porosity, the surface area of the foam increases, resulting in more heat being transferred to the foam.

At a certain frequency, the stiffness with a 60 PPI foam exceeds that of the 40 PPI foam. This behavior is unexpected and requires further investigation. The data points for the 80 PPI foam show the most favorable results in terms of reducing the stiffness of the hydraulic accumulator. The results demonstrate that a reduction in stiffness can be achieved without changing the accumulator's size. According to Hartig [16], a plateau in the mid-frequency range can be observed before the stiffness increases at higher frequencies. At lower frequencies, the state change can be described as isothermal, with heat being transferred to the ambient environment through the accumulator wall. As the frequency increases beyond a certain threshold, the state change becomes adiabatic, and heat exchange occurs only between the gas and the foam via the large surface area of the porous material.



**Fig. 8** Results from the measurement and evaluation of the dimensionless stiffness are presented. The error bars are calculated according to eq. (13) and eq. (14) for the magnitude and phase, respectively. For both porosities of 60 PPI and 80 PPI, the stiffness shows a similar shape.

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## Chapter 10

# Isolating and Quantifying Uncertainties in the Vibration Isolation Round-Robin Challenge

Rileigh Bandy, Rachel Washington, Rebecca Morrison, and Teresa Portone

**Abstract** In the field of structural dynamics, mathematical models generate predictions and inform decisions about complex systems. For example, models may define a bridge's maximum allowable load, chronicle the dynamics of a jointed structure, or describe a driving car's vertical mechanical behavior. But significant discrepancies between observations and predictions often result from model-form uncertainty (e.g., missing dependencies or simplified forces). The Uncertainty Quantification in Vibration Isolation round-robin challenge considers modeling a vibration isolation experiment and the uncertainty that results from a simplified representation of the experimental configuration. As a result of model-form uncertainty, discrepancies emerge between experimental data and model predictions from a linear two-mass oscillator. Last year, we embedded a nonlinear, stochastic enrichment operator into the two-mass oscillator to correct the active damping force. This improved the time-domain amplitude of the interior mass's acceleration, but discrepancies in its time-domain frequency remained. This year, we focus on isolating and quantifying the measurement uncertainty in the experiment, uncertainties in the initial condition, and parameter uncertainties in the linear two-mass oscillator. After Bayesian calibration, the linear two-mass oscillator accurately predicts the resonant frequency of the active vibration isolation experiment.

**Keywords** Measurement uncertainty · Parameter uncertainty · Model discrepancy · Bayesian calibration · Mechanical oscillations

## Introduction

Mathematical models are fundamental for informing predictions and decision making in structural dynamics. For example, models can determine the maximum allowable load of a bridge [1] or express the shear stress of two metal sheets bolted together [2]. However, models do not perfectly represent reality; they include uncertainties that must be quantified before making reliable predictions. These uncertainties accumulate from many sources including uncertain initial conditions or model parameters where accurate values cannot be determined *a priori*. Model-form uncertainties occur due to simplifying assumptions or missing dependencies (e.g., approximating a nonlinear force with a linear empirical function). Additionally, measurement uncertainties are the difference between observed values and reality, and they can result from imprecise measurement sensors.

The purpose of the Uncertainty Quantification in Vibration Isolation round-robin challenge (UQVI-challenge) investigated herein is to question how uncertainties arise and can be addressed when using simple, mathematical models to describe more complex system configurations [3]. Experiments explore the movement of a mass inside a frame after the frame is struck by a modal hammer, with passive and active damping applied [3, 4]. The experimental testing environment provides high-fidelity acceleration data for several combinations of system stiffness, passive and active damping, interior mass values, and high-pass filters. We explore three combinations of varied interior mass values with constant stiffness active damping,

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Rileigh Bandy · Teresa Portone

Optimization and Uncertainty Quantification Department, Sandia National Laboratories, Albuquerque, NM 87123

e-mail: [rjbandy@sandia.gov](mailto:rjbandy@sandia.gov); [tporton@sandia.gov](mailto:tporton@sandia.gov)

Rachel Washington · Rebecca Morrison

Department of Computer Science, University of Colorado, Boulder, CO 80309

e-mail: [Rachel.Washington@colorado.edu](mailto:Rachel.Washington@colorado.edu); [rebeccam@colorado.edu](mailto:rebeccam@colorado.edu)

passive damping, and high-pass filtering. Without quantifying uncertainties, discrepancies emerge between experimental observations and model predictions [5]. Notably, we found in [5] a linear two-mass oscillator model underestimates our quantity of interest (QoI), which is the resonant frequency and its magnitude, an important consequence of the force the hammer inflicts on the interior mass.

Given experimental observations and the model, there are many approaches to estimate, reduce, and propagate uncertainties. One common approach is adding a nonintrusive, stochastic term to the model output [6, 7]. Experimental observations calibrate the stochastic term and result in a catch-all uncertainty representation that can include measurement, initial condition, parameter, and model-form uncertainties. This complicates quantifying the model-form uncertainty because it cannot be isolated.

It is necessary to quantify uncertainties separately and at their source to accurately measure their impact on predictions and to represent their intrinsic nature (e.g., model-form uncertainty originates in the right-hand-side (RHS) of the model, so it should be addressed there too) [8]. Last year, we improved upon the linear two-mass oscillator by embedding nonlinear terms into the RHS of the model to address model-form uncertainty. However, we did not consider parameter and measurement uncertainties or their impact on our representation of model-form uncertainty.

Now, we take a step back to explore if discrepancies can be clearly attributed to uncertain measurements, initial conditions, or model parameters. To address uncertainty in isolation, we focus on first quantifying measurement uncertainties in the experiment and initial condition and parameter uncertainties in the linear two-mass oscillator. We represent uncertainties using the experimental data from the middle interior mass value, and we extrapolate to scenarios with smaller and larger masses. Ultimately, we would like to determine if the linear two-mass oscillator is a sufficient representation of experimental resonant frequencies and corresponding magnitudes.

The rest of the paper is organized as follows. First, three possible distributions for the measurement uncertainty are defined. Next, we describe the linear two-mass oscillator model and plausible distributions for the initial condition and model parameter uncertainties. We detail the methodology how to generate observations, model outputs, and predictions, and Bayesian calibration details. Finally, we present our numerical results and give some concluding remarks.

## Measurement Uncertainty

Before introducing a mathematical model, we define uncertainties in the experimental data. After hitting the frame with a modal hammer, the acceleration of the interior mass over time is measured using an accelerometer. These measurements are uncertain due to sensor precision and variability in the modal hammer hit (e.g., hammer force and strike location). Then, a high-pass filter of 10 Hz preprocesses the acceleration data and removes low-frequency noise [3]. The resulting experimental acceleration data collected from a single hit is

$$\mathbf{a}_{\text{hit}} = a_{\text{hit}}(t) \quad \forall t \in [0, 0.0002, \dots, 12.8], \quad (1)$$

where  $\mathbf{a}_{\text{hit}}$  is a vector of the accelerations ( $\text{m/s}^2$ ) from  $t = 0$  seconds when the modal hammer hits the frame to the final time measured  $t = 12.8$  seconds. Acceleration data is averaged over five sequential hits

$$\bar{\mathbf{a}} = \bar{a}(t) = \frac{1}{5} \sum_{i=1}^5 a_{\text{hit}}^{(i)}(t) \quad \forall t \in [0, 0.0002, \dots, 12.8], \quad (2)$$

which mitigates hit uncertainty, but since five is a relatively small number of samples, hit uncertainty is not entirely removed.

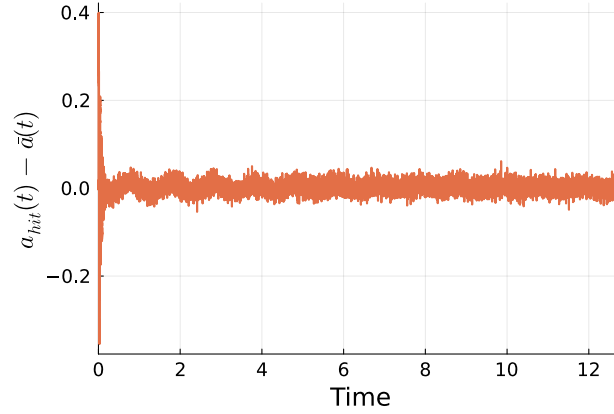
The uncertainties in sensor precision and hits accumulate into the overall measurement uncertainty that we quantify with a distribution informed by the experimental data. To inform our experimental uncertainty model, we have access to  $\bar{a}(t)$  and the final hit of five hits used to compute the mean. There are several possible formulations to represent the experimental uncertainty. We investigate three options: additive Gaussian, additive Gaussian with a smaller variance, and a mixture of multiplicative and additive Gaussian distributions.

**Additive Gaussian Measurement Uncertainty:** Additive Gaussian distributions are commonly used to model measurement uncertainty and are commonly referred to as additive white Gaussian noise [9]. As the name implies, measurement uncertainty or noise is added to the measurement. For a single hit, we assume the measured acceleration of the interior mass at time  $t$  is independent. Assuming independence imposes less structure than dependence, and it would be challenging to elicit correlations in time from the current data. (However, assuming measurements are independent in time could overvalue the data in forthcoming Bayesian calibration.) Therefore, for a single hit, we assume the measurement uncertainty is additive Gaussian as such,

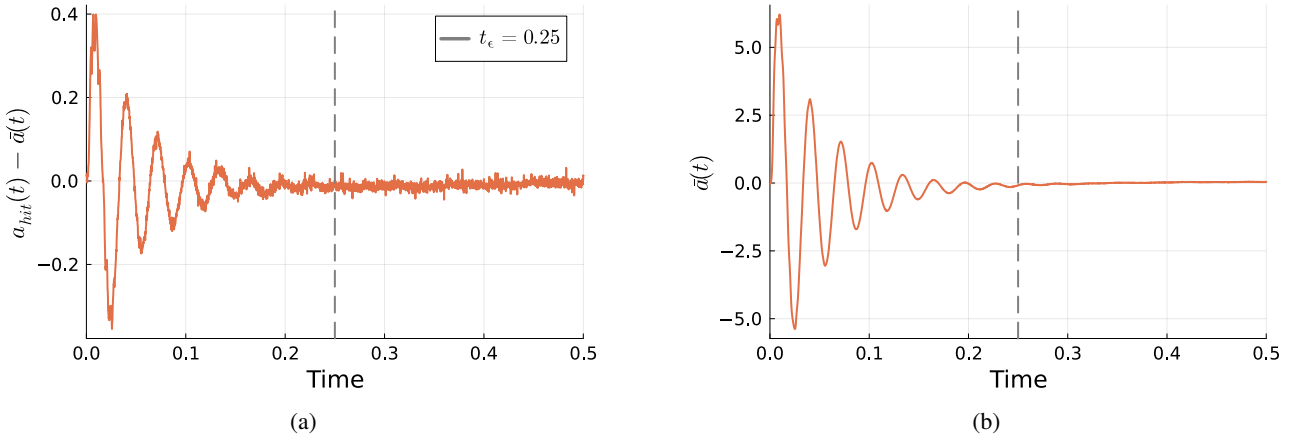
$$a_{\text{hit}}(t) = \bar{a}(t) + \epsilon \sim \mathcal{N}(\bar{a}(t), \sigma_{\epsilon}^2). \quad (3)$$

We explore two different values for  $\sigma_\epsilon^2$ :  $\sigma_\epsilon^2 = 0.05$  (about 1% of the maximum acceleration observed in the experimental data) or  $\sigma_\epsilon^2 = 0.009^2 = 8 \times 10^{-5}$  (a much smaller variance defined later to capture sensor uncertainty). The additive Gaussian distribution is not based on uncertainty information from the experiment. Therefore, we define an alternative distribution using more information.

**Mixture Model Measurement Uncertainty:** Given the mean acceleration data and the acceleration data from the final hit, we investigate the deviation of  $a_{\text{hit}}(t)$  from  $\bar{a}(t)$ , shown in Figure 1. Initially the mean deviations vary depending on time, but they quickly become stable. To investigate where this switch occurs, we zoom in on the first 0.5 seconds in Figure 2a. Mean deviations consistently fall within roughly  $[-0.05, 0.05]$  after  $t = 0.15$ . We also investigate the mean acceleration data for the first 0.5 seconds in Figure 2b, and the mean acceleration data appears to reach a stationary mode around  $t = 0.25$ . Therefore, we select  $t_\epsilon = 0.25$  as the time when measurement uncertainty switches from time-varying to constant. We believe this is where sensor uncertainty dominates and hit uncertainty becomes less influential for the overall measurement uncertainty.



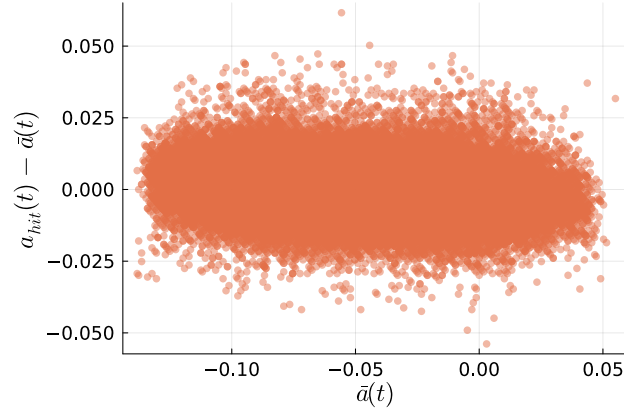
**Fig. 1** Experimental acceleration mean deviations from the full time-domain.



**Fig. 2** Selection of the time where measurement uncertainty goes from time-varying to constant  $t_\epsilon$  based on (a) experimental acceleration mean deviations and (b) mean experimental acceleration.

Given the shift from time-varying to constant mean deviations at  $t_\epsilon$ , we define the measurement uncertainty in two parts, with a distribution for hit uncertainty and a distribution for sensor uncertainty. To quantify sensor uncertainty, we investigate the relationship between mean acceleration and mean deviations for times after  $t_\epsilon$  in Figure 3. The mean deviations are independent of mean acceleration; they appear to be Gaussian distributed with a mean of zero and a sample standard deviation of 0.009. We assume the final hit falls within a 50% confidence interval of our sensor distribution making the distribution's standard deviation twice as large. Therefore, we characterize sensor uncertainty as

$$\epsilon_{\text{sensor}} \sim \mathcal{N}(0, 0.018^2). \quad (4)$$

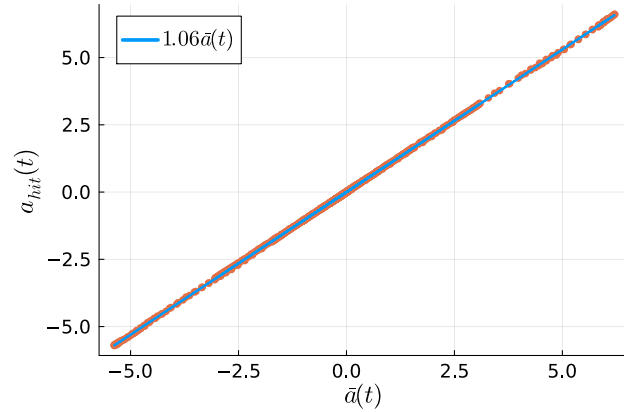


**Fig. 3** Experimental acceleration mean deviations after  $t_\epsilon$ .

To quantify hit uncertainty, we investigate the relationship between mean acceleration and the final hit for times before  $t_\epsilon$  in Figure 4. Final hit acceleration is linearly dependent on mean acceleration. That dependence is characterized with the linear function  $1.06\bar{a}(t)$ , where the slope is estimated using the maximum and minimum final hit accelerations and their associated mean accelerations. Similar to sensor uncertainty, we approximate hit uncertainty standard deviation with twice the sample standard deviation of acceleration mean deviations, which is 0.1.

Therefore, we assume hit uncertainty is multiplicative, Gaussian and characterize it as

$$\epsilon_{\text{hit}} \sim \mathcal{N}(1.06, 0.2^2). \quad (5)$$



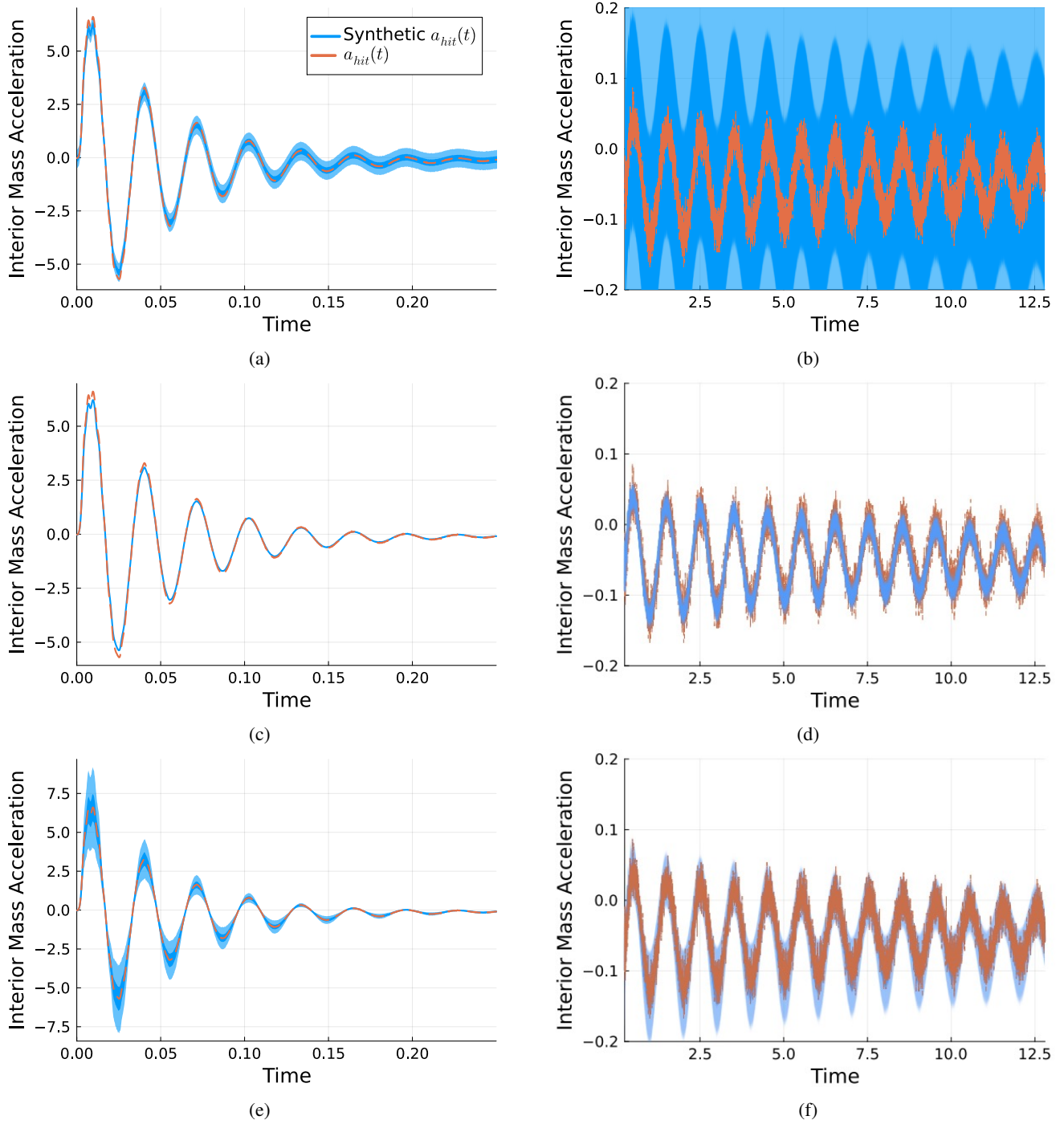
**Fig. 4** Relationship between mean experimental acceleration and final hit experimental acceleration before  $t_\epsilon$ .

After characterizing sensor and hit uncertainties, we bring them together to form our measurement uncertainty mixture model:

$$a_{\text{hit}}(t) = \epsilon_{\text{hit}}\bar{a}(t) + \epsilon_{\text{sensor}} \sim \mathcal{N}(1.06\bar{a}(t), 0.2^2\bar{a}(t)^2 + 0.018^2). \quad (6)$$

**Measurement Uncertainty Pushforward:** Could the acceleration data from the final hit plausibly be drawn from any of the three distributions of measurement uncertainty defined above? To answer this question, we use mean experimental acceleration data and Equations (3) and (6) to generate 5000 synthetic hits and compare them to acceleration data from the final hit. In Figure 5 the resulting 50% and 95% confidence intervals from the synthetic hits are depicted as the dark and lighter blue shaded regions.

Additive Gaussian measurement uncertainty with  $\sigma_\epsilon^2 = 0.05$  is pushed forward in Figures 5a and 5b, where Figure 5a shows trajectories until 0.25 seconds, and Figure 5b shows trajectories from 0.25 seconds to 12.8 seconds. This measurement uncertainty distribution encompasses the final hit's trajectory throughout time, but it could be overestimating uncertainties as acceleration dampens and uncertainty remains constant. Since  $a_{\text{hit}}(t)$  is encompassed within the confidence region, the experimental acceleration data could be plausibly drawn from this distribution of measurement uncertainty.



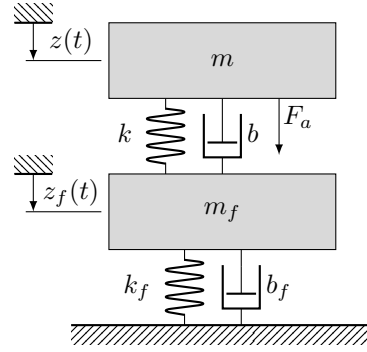
**Fig. 5** Comparison of experimental acceleration data to the 50% and 95% confidence intervals (dark and lighter blue shaded regions, respectively) from synthetic data generated from three measurement uncertainty distribution: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b), additive Gaussian with  $\sigma_\epsilon^2 = 8 \times 10^{-5}$  in (c, d), and the mixture model in (e, f). Subfigures (a,c,e) depict times before  $t_\epsilon$  and (b,d,f) depict times after  $t_\epsilon$ .

Additive Gaussian measurement uncertainty with  $\sigma_\epsilon^2 = 8 \times 10^{-5}$  is pushed forward in Figures 5c and 5d. This measurement uncertainty distribution noticeably underpredicts the first three peaks and troughs compared to the final hit's acceleration, and it seems to slightly underpredict the uncertainties as acceleration dampens. Therefore, experimental acceleration data could not be plausibly drawn from this distribution of measurement uncertainty.

The mixture model measurement uncertainty is pushed forward in Figures 5e and 5f. In Figure 5e the 50% confidence interval encompasses experimental acceleration data with the most spread at the initial peaks and troughs. The spread decreases as time progresses and acceleration dampens. In Figure 5f the 95% confidence interval closely matches and encompasses most of the experimental acceleration data. Therefore, the experimental acceleration data could be plausibly drawn from this distribution of measurement uncertainty, and it is the most plausible candidate of the three measurement uncertainties. To show the impact of plausible and implausible measurement uncertainty on a mathematical model's predictions, we propagate uncertainties using all three distributions.

## Parameter and Initial Condition Uncertainties

Now that we have defined three possible measurement uncertainty distributions, we can introduce a mathematical model and start quantifying its uncertainties. We start with a linear two-mass model of the experimental setup shown in Figure 6, where one mass represents the interior mass and one mass approximates the frame itself. The displacement of the interior mass is represented by  $z(t)$ , the displacement of the frame is  $z_f(t)$ , and both are measured in meters.



**Fig. 6** Mass-spring-damper diagram of the two-mass oscillator.

The accelerations of the linear two-mass oscillator are defined as

$$\ddot{z}(t) = \frac{1}{m} \{-b(\dot{z}(t) - \dot{z}_f(t)) - g\dot{z}(t) - k(z(t) - z_f(t))\} \quad (7)$$

$$\ddot{z}_f(t) = \frac{1}{m_f} \{b(\dot{z}(t) - \dot{z}_f(t)) + k(z(t) - z_f(t)) - b_f\dot{z}_f - k_f z_f\}. \quad (8)$$

The acceleration of the interior mass ( $m$ ) is defined by  $\ddot{z}(t)$ , which depends on the passive damping coefficient  $b$ , the active damping coefficient  $g$ , and the spring constant  $k$ . The acceleration of the frame mass ( $m_f$ ) is defined by  $\ddot{z}_f(t)$ , which also depends on the frame's damping coefficient  $b_f$  and spring constant  $k_f$ . The passive damping coefficients are

$$b = 2D\sqrt{\frac{k}{m}} \quad (9)$$

$$b_f = 2D_f\sqrt{\frac{k_f}{m_f}}, \quad (10)$$

where  $D$  and  $D_f$  are the passive damping ratios.

The linear two-mass oscillator only considers damping and spring forces, so there is not an analogous impulse force to the modal hammer hit from the experiment. Instead, the linear two-mass oscillator is integrated forward in time starting after the initial impulse force subsides, which we estimate is at  $T_i = 0.01$  seconds. Therefore, the initial conditions for the linear two-mass oscillator  $\mathbf{Z} = (z(T_i), z_f(T_i), \dot{z}(T_i), \dot{z}_f(T_i))$  are uncertain, and we must approximate them with a distribution using prior information. We define the prior distribution on the initial conditions as

$$p(\mathbf{Z}) = \prod_{i=1}^4 p(Z_i) = \prod_{i=1}^4 \mathcal{U}(-0.5, 0.5). \quad (11)$$

The initial condition priors are maximum entropy priors, where the distribution with the highest entropy that fits a given set of constraints is selected [10]. Here, the maximum entropy priors are uniform, since we expect the initial displacements and velocities to be small and between roughly  $-0.5$  and  $0.5$  in order for the initial interior mass acceleration to be around  $5 \text{ m/s}^2$ .

The model parameters of the linear two-mass oscillator  $\theta = (k, D, g, k_f, D_f)$  are also uncertain. The nominal model parameters  $\theta^0 = (k^0, D^0, g^0, k_f^0, D_f^0)$  are derived from the experimental apparatus before gathering data, and they are shown in Table 1. For example, the interior mass's spring constant is derived from the assumed bending stiffness  $EI$  and length  $l$ ,  $k^0 \propto \frac{EI}{l^3}$ . While the nominal values give us good initial estimates, they are uncertain and could be improved using experimental data. Therefore, we approximate them with a distribution using prior information. We define the prior distribution on the model parameters as

$$p(\theta) = \prod_{i=1}^5 p(\theta_i) = \prod_{i=1}^5 \mathcal{N}(\theta_i^0, 0.2\theta_i^0). \quad (12)$$

The model parameters assume maximum entropy priors centered at their nominal values with standard deviations of 20% of those nominal values.

**Table 1** Nominal parameter values derived from the experimental apparatus before gathering data.

| Parameter |       |      |
|-----------|-------|------|
| $k^0$     | 32583 | N/m  |
| $D^0$     | 4.85  | %    |
| $g^0$     | 26    | Ns/m |
| $k_f^0$   | 722   | N/m  |
| $D_f^0$   | 0.1   | %    |

## Methodology

After isolating and quantifying the uncertainties in measurements, initial conditions, and parameters, we must determine whether the linear two-mass oscillator adequately represents the experiment. To address this, we perform Bayesian calibration using post-processed observations from the experiment.

**Observations:** During the round-robin experiments, five sequential hits were conducted, and the results were filtered using a high-pass analogue filter with a cutoff frequency of 10 Hz and then averaged [4]. Using this averaged and filtered data, we interpolated the time series,  $\bar{a}$ , to obtain discrete data at uniform time intervals. The resulting observations are

$$D(X) = \text{interpolate}(\bar{a}(t; X)), \quad \forall t \in (T_i, T_i + tstep, \dots, 12.8), \quad (13)$$

where the experiment is initialized with masses

$$X = (m, m_f). \quad (14)$$

The time after the initial impulse force is  $T_i = 0.01$  seconds and the timestep between discrete observations is  $tstep = 0.001$  seconds. We subsample calibration data from one set of masses for a short time-window to create calibration observations, defined as

$$\begin{aligned} D_c \subset D(X_c) &= D(m = 0.9249 \text{ kg}, m_f = 9.33 \text{ kg}) \\ D_c &= \text{interpolate}(\bar{a}(t; X_c)), \quad \forall t \in (T_i, T_i + tstep, \dots, 0.072). \end{aligned} \quad (15)$$

This short time-window is selected to include the first two wavelengths after the initial impulse force subsides.

We subsample validation data to include an interior mass both larger and smaller than the calibration scenario, while the frame mass remains the same:

$$D_v = \{D(X_{v_1}), D(X_{v_2})\} = \{D(m = 0.7853 \text{ kg}, m_f = 9.33 \text{ kg}), D(m = 1.0613 \text{ kg}, m_f = 9.33 \text{ kg})\}. \quad (16)$$

**Model Output:** The linear two-mass oscillator simulates the interior mass's movement; we then sample the trajectory of the interior mass's acceleration at discrete intervals in time. This results in the following output:

$$Y(\theta, Z, X) = \ddot{z}(t; \theta, Z, X), \quad \forall t \in (T_i, T_i + tstep, \dots, 12.8), \quad (17)$$



where  $\theta$  are model parameters and  $Z$  are initial conditions. Outputs for the calibration scenario are defined as

$$\begin{aligned} Y_c(\theta, Z) &\subset Y(\theta, Z, X_c) \\ Y_c(\theta, Z) &= \ddot{z}(t; \theta, Z, X_c), \quad \forall t \in (T_i, T_i + tstep, \dots, 0.072), \end{aligned} \quad (18)$$

and outputs for the validation scenarios are defined as

$$Y_v(\theta, Z) = \{Y(\theta, Z, X_{v_1}), Y(\theta, Z, X_{v_2})\}. \quad (19)$$

**Model Predictions of the QoI:** Acceleration of the interior mass can also be analyzed in the frequency domain. We transform observations and model outputs using the discrete Fourier transform (DFT):

$$F(\theta, Z, X) = |\text{DFT}(Y(\theta, Z, X))|, \quad (20)$$

where the frequencies above the Nyquist frequency are removed [11]. The time domain (in seconds) is transformed to the frequency domain (in Hz), and the interior mass's acceleration (in  $\text{m/s}^2$ ) is transformed to the absolute magnitude (in  $\text{m/s}^2$ ). We use the time-domain from  $T_i$  until 12.8 seconds to improve the resolution of the frequency-domain data. Since the linear two-mass oscillator model has two degrees of freedom, there are two modes of vibration and two frequencies where the modes occur (resonant frequencies). The lower resonant frequency ( $\approx 2\pi$  Hz) corresponds to the frame, and the higher resonant frequency ( $\approx 10^{1.5}$  Hz) corresponds to the interior mass [4]. We omit the frame's resonant frequency, and our QoI is the interior mass's resonant frequency and associated magnitude:

$$\text{QoI} = (\alpha, \lambda^*) = \left( \max_{\lambda \in \Lambda} \{F(\theta, Z, X; \lambda)\}, \arg\max_{\lambda \in \Lambda} \{F(\theta, Z, X; \lambda)\} \right), \quad (21)$$

where  $\Lambda = (10, 100)$  Hz. This QoI is important for quantifying the interior mass's response to forcing.

The QoI prediction for the calibration scenario is defined as

$$\text{QoI}_c = (\alpha_c, \lambda_c^*) = \left( \max_{\lambda \in \Lambda} \{F(\theta, Z, X_c; \lambda)\}, \arg\max_{\lambda \in \Lambda} \{F(\theta, Z, X_c; \lambda)\} \right), \quad (22)$$

and QoIs for the validation scenarios are defined as

$$\begin{aligned} \text{QoI}_{v_1} &= (\alpha_{v_1}, \lambda_{v_1}^*) = \left( \max_{\lambda \in \Lambda} \{F(\theta, Z, X_{v_1}; \lambda)\}, \arg\max_{\lambda \in \Lambda} \{F(\theta, Z, X_{v_1}; \lambda)\} \right) \\ \text{QoI}_{v_2} &= (\alpha_{v_2}, \lambda_{v_2}^*) = \left( \max_{\lambda \in \Lambda} \{F(\theta, Z, X_{v_2}; \lambda)\}, \arg\max_{\lambda \in \Lambda} \{F(\theta, Z, X_{v_2}; \lambda)\} \right). \end{aligned} \quad (23)$$

**Bayesian Calibration:** We propagate our uncertainties to the model output using Bayesian calibration. The posterior distribution of the model parameters and initial conditions is sampled using the Adaptive Scaling within Adaptive Metropolis (ASWAM) method in Julia [12, 13]. The posterior is

$$\pi_{post} = p(\theta, Z \mid D_c) \propto \pi_{like} \pi_{pri}. \quad (24)$$

The prior distribution is the product of the initial condition and model parameter priors:

$$\pi_{pri} = p(\theta)p(Z). \quad (25)$$

The likelihood distribution measures how well the model explains experimental data given the measurement uncertainty. Therefore, we explore three possible likelihood distributions:

$$\pi_{like} = p(D_c \mid \theta, Z) = \begin{cases} \mathcal{N}(Y_c(\theta, Z); D_c, 0.05I) & \text{(additive Gaussian)} \\ \mathcal{N}(Y_c(\theta, Z); D_c, 8 \times 10^{-5}I) & \text{(small additive Gaussian)} \\ \mathcal{N}(Y_c(\theta, Z); 1.06D_c, \text{diag}(0.2^2 D_c^2) + 0.018^2 I) & \text{(mixture model)} \end{cases} \quad (26)$$

following the measurement uncertainties defined in Equations (3) and (6).



Calibration took 300,000 Markov Chain Monte Carlo (MCMC) steps with five parallel tempering levels. In adaptive parallel tempering, each level had an independent random walk sampler, and the first level was randomly swapped with an auxiliary level (levels two through five) in order to reach an average switching probability of 0.234 [12]. The first 50,000 MCMC steps were thrown out as burnin, and the lag was 50 MCMC steps. Each calibration took under ten minutes on a single Apple M3 Max CPU core, and data was stored on an SSD. We assessed convergence using trace plots, where each initial condition and parameter value is plotted against filtered MCMC steps. We examined the trace plots for visual patterns, where no major trends or jumps occurred indicating that the chains likely reached a stationary distribution [14].

While calibration with all three likelihoods appears to have converged, calibration using the small additive Gaussian likelihood was the most challenging. A narrower range in initial conditions and model parameters was sampled, and sometimes that range was limited by the priors. For example, with small additive Gaussian likelihood, samples of the active damping coefficient  $g$  ranged from 36 to 40, which were roughly two to three standard deviations larger than the prior mean of 26. With the mixture model likelihood, samples of  $g$  ranged from 15 to 30.

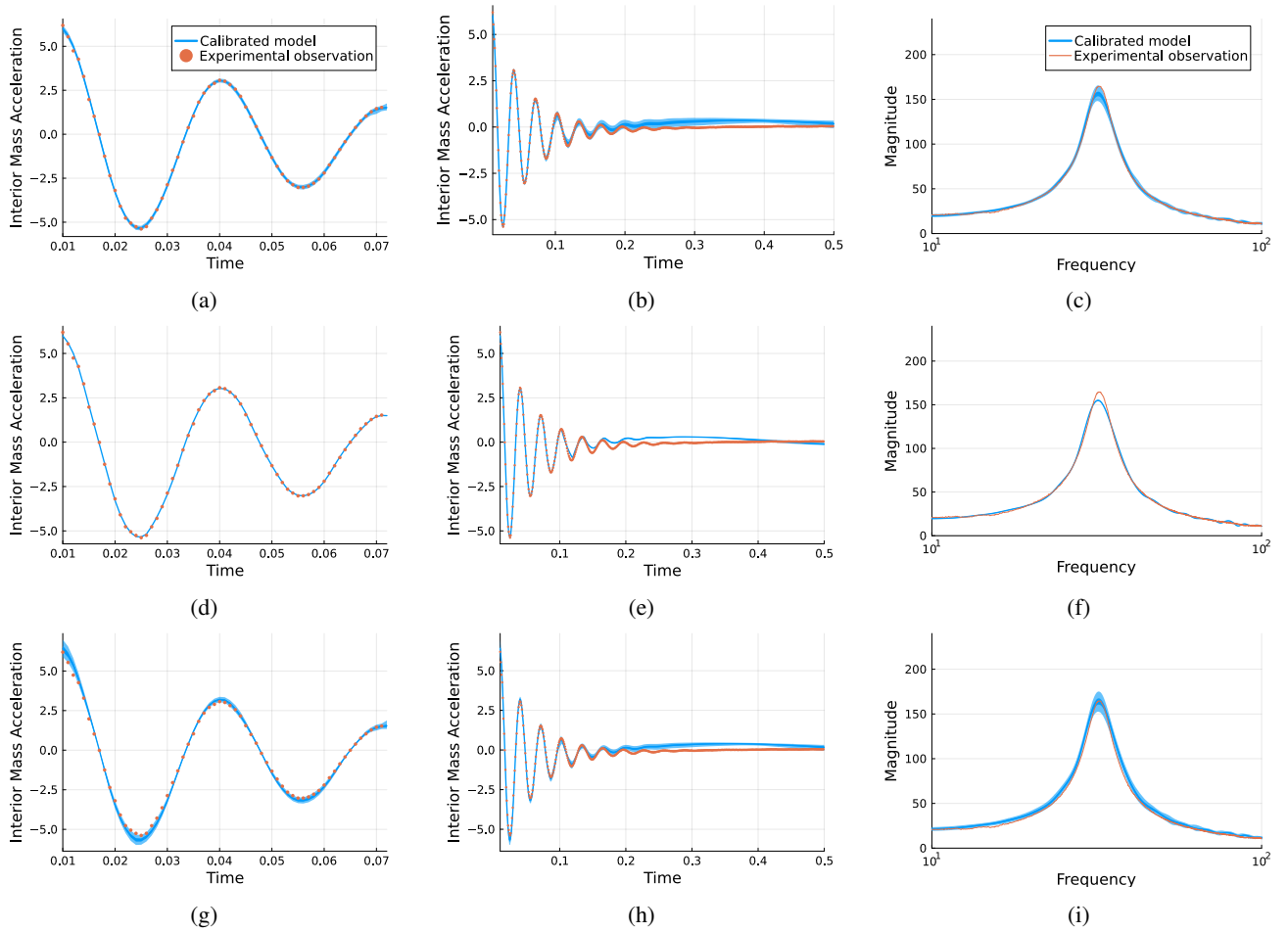
## Numerical Results

After calibrating the linear two-mass oscillator, uncertainties propagate to model outputs, transformed outputs in the frequency-domain, and QoI predictions. Numerical results from the calibrated model with the three likelihood formulations are compared to experimental observations for the calibration scenario and two validation scenarios. For the validation scenarios, we update the interior mass, but do not recalibrate the initial conditions or model parameters. We omit the frame mass's data during calibration and validation because we anticipate it introducing additionally, unresolved model-form uncertainties [5].

**Calibration Scenario:** Numerical results are compared to experimental observations for the calibration scenario in Figures 7a, 7d and 7g, where the calibrated model closely matches experimental observations for all three likelihood distributions. However, the likelihood directly influences the calibrated model confidence intervals. For example, the small additive Gaussian likelihood in Figure 7d results in narrow confidence intervals in the output, where 50% and 95% confidence intervals are visually indistinguishable. When the mixture model likelihood is used in Figure 7g, model output is most uncertain at the first few peaks and troughs, which tracks with its measurement uncertainty pushforward in Figure 5e. When the time-frame is extended in Figures 7b, 7e and 7h, discrepancies emerge after the first couple oscillations and persist in time. Notably, with the small additive Gaussian likelihood in Figure 7e, discrepancies are larger and appear earlier than in Figures 7b and 7h. Time-domain trajectories are transformed to the frequency-domain in Figures 7c, 7f and 7i. The calibrated models with additive Gaussian likelihood and mixture model likelihood encompass most experimental observations, while the calibrated model with small additive Gaussian likelihood underpredicts the resonant frequency's magnitude. Since the resonant frequency and its magnitude are our QoI, it is important for the calibrated model to predict them well.

We explore the calibration scenario QoI predictions in Figure 8, where the marginal calibrated model distributions of the resonant frequency and its magnitude are compared to the experiment. The marginals of the calibrated models with additive Gaussian likelihood and mixture model likelihood encompass the experimentally observed resonant frequency and maximum magnitude. The marginals from the mixture model likelihood have higher density around the experiment, indicating that the experimentally observed QoI was more likely to be generated from the model with mixture model likelihood than the model with additive Gaussian likelihood. The experimentally observed maximum magnitude falls outside of the marginal distribution of the calibrated model with small additive Gaussian likelihood in Figure 8d. Therefore, we deem the linear two-mass oscillator with small additive Gaussian likelihood insufficient for predicting the QoI of the experiment. Because the linear two-mass oscillator with small additive Gaussian likelihood is not predictive in the calibration scenario, we do not extend its predictions to the validation scenarios.

We investigated extending the time-frame of the calibration scenario to incentivize model output to be closer to experimental observations, but Bayesian calibration would not converge. This could be caused by correlation of the measurement uncertainties overtime, which we assumed to be independent. Increasing the time-frame of the calibration data caused the importance of the likelihood to increase. However, the likelihoods also became more restrictive. Specifically, the likelihoods did not allow for much deviation in the time-domain frequency. Many parameter and initial condition combinations of the model resulted in a frequency shift that the likelihoods determined improbable. Therefore, our MCMC chains would not mix and oscillated between a few parameter and initial condition combinations regardless of the number of parallel levels we added, the number of steps we took, or even the MCMC sampling methods used.



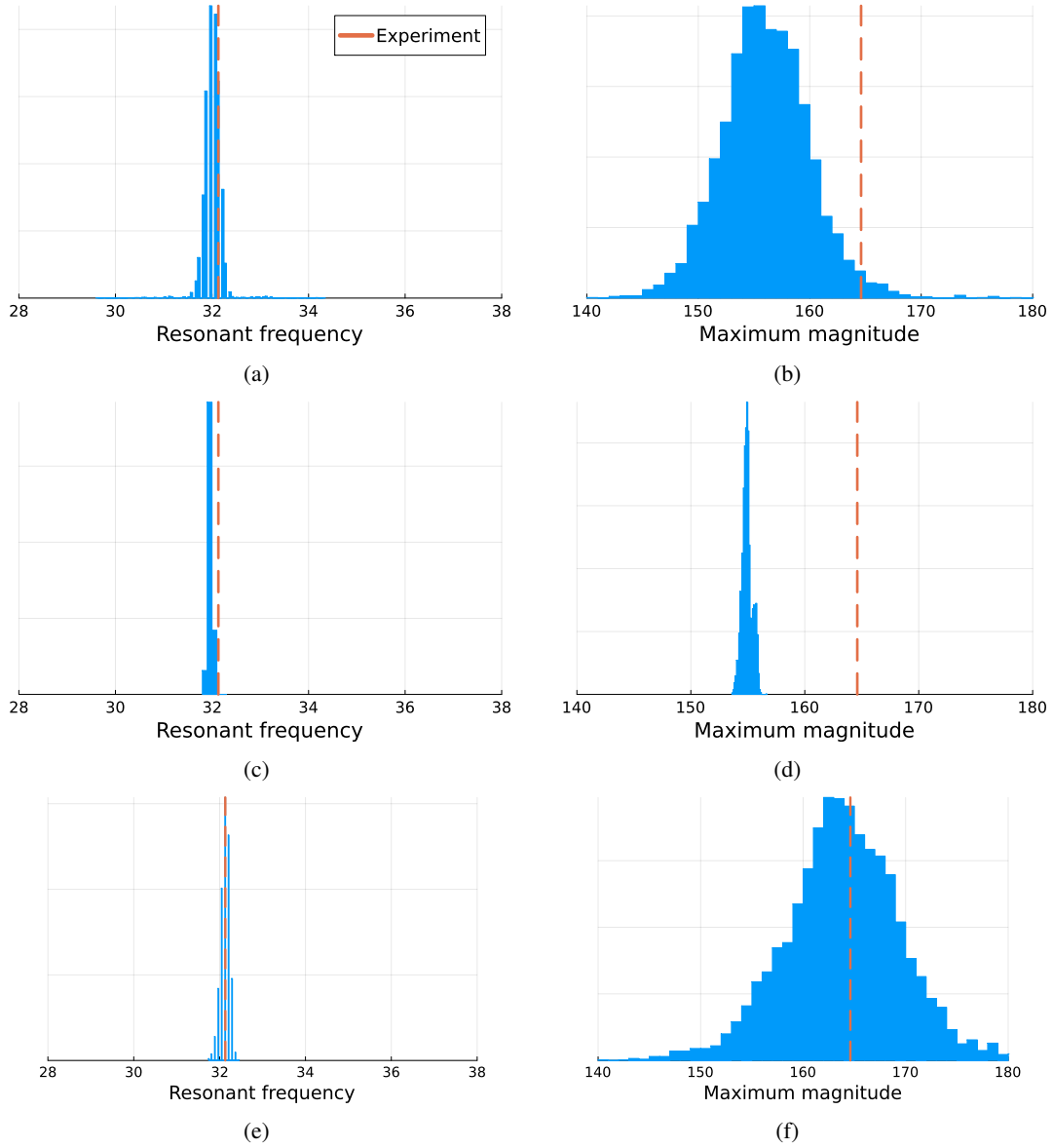
**Fig. 7** Comparison of experimental observations to the 50% and 95% confidence intervals (dark and lighter blue shaded regions) from the calibrated model with three likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b, c), additive Gaussian with  $\sigma_\epsilon^2 = 8 \times 10^{-5}$  in (d, e, f), and the mixture model in (g, h, i) for (a, d, g) the calibration time-window, (b, e, h) extended times, and (c, f, i) the frequency-domain.

We hypothesize additional, unresolved uncertainties caused Bayesian calibration of the extended time-frame to fail. Notably, and in contrast to our work last year, we did not resolve model-form uncertainties in the linear two-mass oscillator. For example, we approximate the frame of the experiment as a mass, and we assume there are only linear spring and damping forces acting on the system. In the future, we would like to explore these uncertainties further.

**Validation Scenarios:** In each validation scenario, the interior mass is changed, but we do not recalibrate the linear two-mass oscillator. Instead, we propagate samples from the calibrated model's initial conditions and model parameters to model outputs in the validation scenarios. We assume the experimental test setup was not disassembled and reassembled between scenarios, and the only change in the experiment was increasing/decreasing the interior mass. Then, we test to see if the linear two-mass oscillator reliably extrapolates to the validation scenarios.

Numerical results are compared to experimental observations for validation scenario 1 in Figure 9, where the time-domain frequency of the calibrated model is slightly slower than experimental observations for the additive Gaussian likelihood and the mixture model likelihood. In the frequency-domain, predicted resonant frequencies are slightly lower than the experimentally observed resonant frequency. These discrepancies indicate potential unresolved model-form uncertainty since a perfect model of the experiment should be able to extrapolate to unobserved mass values.

We explore the QoI predictions for validation scenario 1 in Figure 10, where the marginal calibrated model distributions of the resonant frequency and maximum magnitude are compared to the experiment. The experimentally observed resonant frequency is in the right tail of both models' marginal distributions, while the experimentally observed magnitude is in a higher density region of the marginal distribution from the mixture model likelihood. Because the marginals



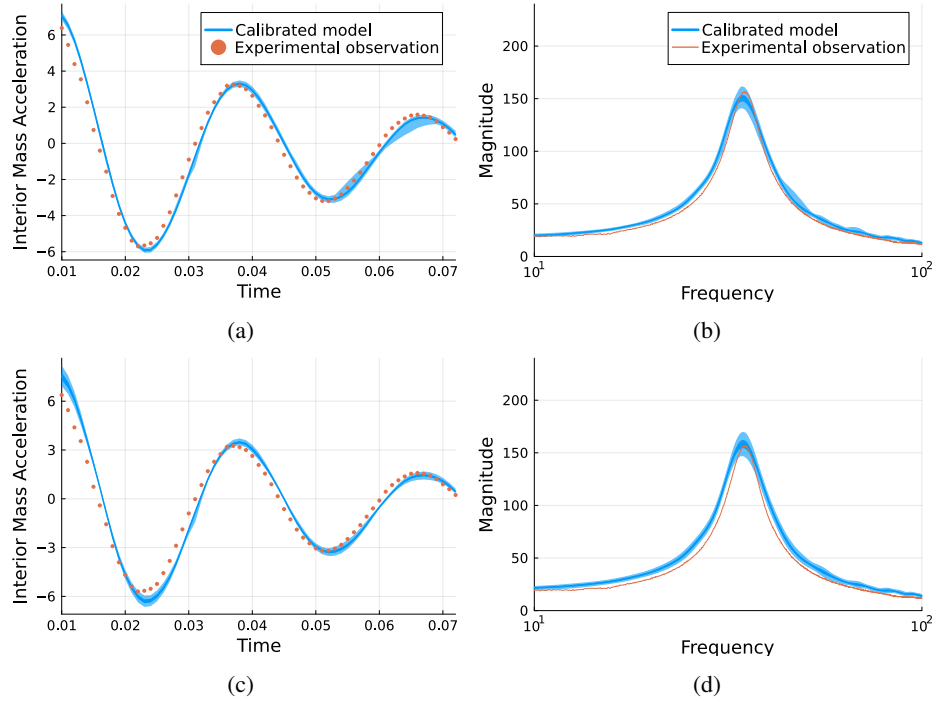
**Fig. 8** Comparison of calibration scenario experimental QoI (resonant frequency and maximum magnitude) to their marginal distributions from the calibrated model with three likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b), additive Gaussian with  $\sigma_\epsilon^2 = 8 \times 10^{-5}$  in (c, d), and the mixture model in (e, f).

from both likelihoods are similar, additional quantitative validation is needed to determine the best model for this validation scenario.

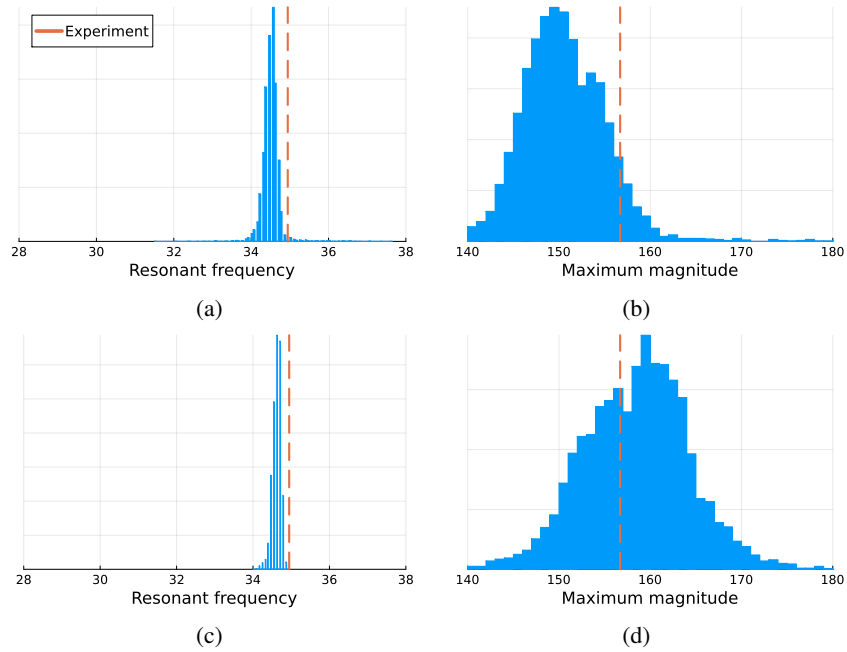
Numerical results are compared to experimental observations for validation scenario 2 in Figure 11, where the time-domain frequency of the calibrated models closely matches experimental observations. In Figure 11b, the first peak/trough are slightly underestimated/overestimated by the additive Gaussian likelihood, while in Figure 11c the calibrated model with a mixture model likelihood is able to encompass more observations in its 95% confidence interval.

Validation scenario 2 QoI predictions are shown in Figure 12. Again, the experimentally observed resonant frequency is in the right tail of both models' marginal distributions. The experimentally observed magnitude is in a higher density region of the marginal distribution from the additive Gaussian likelihood. Quantitative validation is needed to determine the best model for this validation scenario as well.

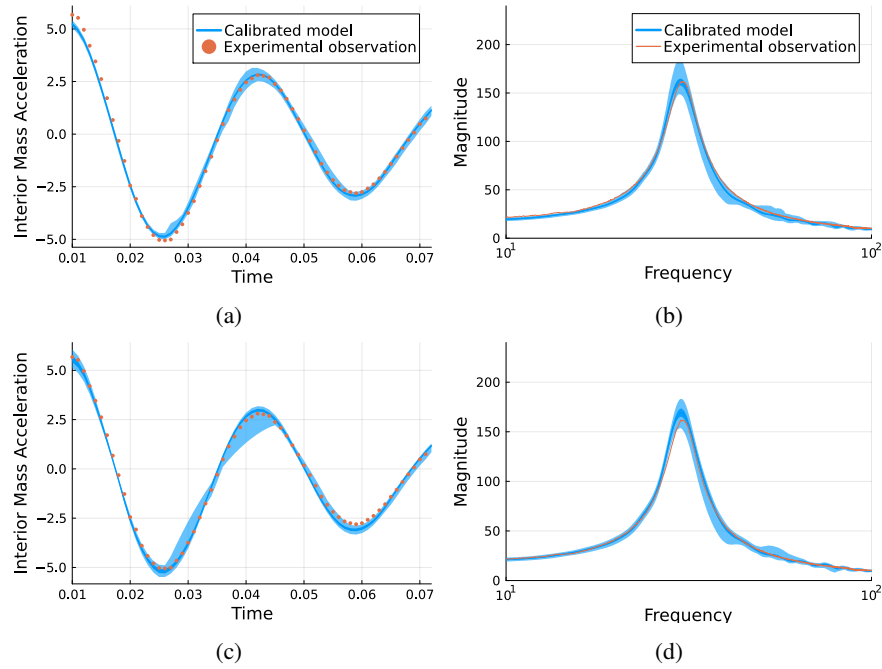
Model validation, which evaluates if a mathematical model is a reliable representation of reality, strongly depends on the QoI. The linear two mass oscillator with either additive Gaussian likelihood or mixture model likelihood are promising



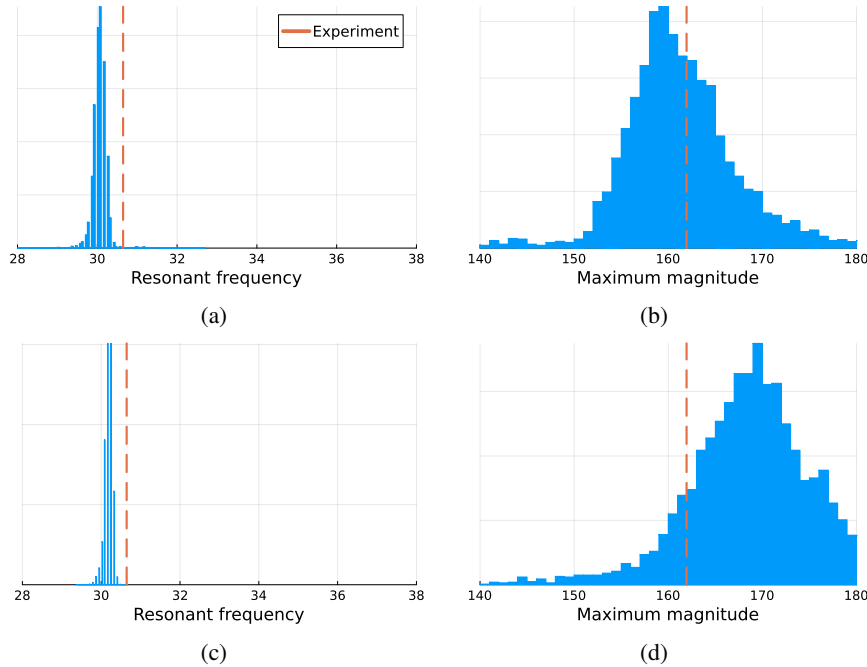
**Fig. 9** Validation scenario 1 ( $m = 0.7853$  kg) comparison of experimental observations to model output 50% and 95% confidence intervals (dark and lighter blue shaded regions) from the calibrated model with two likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b) and the mixture model in (c, d) for (a, c) the calibration time-window and (b, d) the frequency-domain.



**Fig. 10** Comparison of validation scenario 1 experimental QoI (resonant frequency and maximum magnitude) to their marginal distributions from the calibrated model with two likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b) and the mixture model in (c, d).



**Fig. 11** Validation scenario 2 ( $m = 1.0613$  kg) comparison of experimental observations to model output 50% and 95% confidence intervals (dark and lighter blue shaded regions) from the calibrated model with two likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b) and the mixture model in (c, d) for (a, c) the calibration time-window and (b, d) the frequency-domain.



**Fig. 12** Comparison of validation scenario 2 experimental QoI (resonant frequency and maximum magnitude) to their marginal distributions from the calibrated model with two likelihood distributions: additive Gaussian with  $\sigma_\epsilon^2 = 0.05$  in (a, b) and the mixture model in (c, d).

models for predicting the experiment's resonant frequency and maximum magnitude. If the QoI changed to the time-domain acceleration data up to 0.5 seconds, all of these models would be invalidated.

## Conclusion

In this work, we investigate and reduce discrepancies between experiments and the linear two-mass oscillator model caused by uncertainties in measurements, initial conditions and model parameters. We isolate and address each uncertainty at its source, and we inform parameter and initial condition uncertainties using experimental data. After Bayesian calibration, the linear two-mass oscillator with additive Gaussian and mixture model measurement uncertainty accurately predict the resonant frequency and associated magnitude of the active vibration isolation experiment. However, discrepancies in the time-domain could be reduced as trajectories dampen. We hypothesize that model-form uncertainties caused these discrepancies, and in the future, we plan to integrate our work from last year into this approach to quantify model-form uncertainties.

The importance of quantifying measurement uncertainty cannot be overstated and is evident in the strong impact of the likelihood in Bayesian calibration and model predictions. In the future, more work could be done to quantify a robust distribution for the measurement uncertainty. If we had access to multiple individual hits instead of one, we could more directly represent measurement uncertainties rather than modeling them as we did here. For example, we could build a covariance matrix to represent correlations in time instead of assuming independence.

Getting back to a primary motivation of the UQVI-challenge, we show that model validation depends on the QoI. The QoI should be selected based on the use case of the model, since the model may be valid for predicting one QoI but not another as we demonstrated. In order to assess diverse measures in quantifying uncertainties of various mathematical models, we need careful consideration of use cases and rigorous validation with uncertainty quantification to help determine which models are sufficient and to enable selection between multiple candidate models.

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# Chapter 11

## Statistical Framework for Deep Learning Model Comparison and Evaluation



Samuel Myren, Nidhi Parikh, Rosalyn Rael, Garrison Flynn, Dave Higdon, and Emily Casleton

**Abstract** Deep learning for structural health monitoring (SHM) has enabled researchers to extract information from big data acquired from a sensor network or images to detect, identify, and characterize events of interest. However, choosing the best model from a set of candidates for a given problem remains challenging due to the need for a robust evaluation framework that considers model performance and the uncertainty in the employed data environments. We contribute such a framework for deep learning model evaluation and comparison that is rooted in statistical foundations. Given a performance metric of interest and a set of competing models or training approaches, our framework estimates the distribution of performance metrics and segregates uncertainty arising from the training data choice and the model fitting process (i.e. weight initialization and stochastic gradient descent). We demonstrate this framework by training a deep learning architecture using ground vibration data to detect events under a set of training conditions to quantify the effects of pre-training, fine-tuning, and out-of-distribution training approaches. Additionally, we evaluate these models across varying amounts of training data to mimic real world scenarios where data labeling is expensive/limited and the model is deployed in an environment slightly different from the training environment. In this demonstration, we exemplify how easy it can be to select the incorrect model if the models are only trained once and that model initialization can unexpectedly influence performance uncertainty. Our rigorous and generalizable framework which is supported through a real example provides value to the SHM researchers by enabling them to confidently compare various models and rank them according to their priorities.

**Keywords** Uncertainty Quantification · Foundation Models · Learning Efficiency · Artificial Intelligence · Signal Processing

### Introduction

Equipped with technological advances in artificial intelligence (AI) and big data from advanced sensors, engineers and practitioners in structural health monitoring (SHM) are actively researching and incorporating traditional deep learning (DL) models to accomplish tasks within their workflows (Flah et al., 2021). In recent years, these advances have culminated with foundation models (FMs) (Bommasani et al., 2022; Harsuko & Alkhalifah, 2022), large models pre-trained on vast amounts of data that can be fine-tuned using smaller, labeled datasets to accomplish a variety of specific tasks. FMs take traditional DL a step further by promising to accomplish multiple tasks simultaneously, even potentially replacing the workflow altogether. However, for the engineering and scientific domains such as SHM and geosciences, we lack robust evaluation frameworks for even the simpler DL models, which ill-prepares us to evaluate FMs upon their inevitable adoption.

We address this challenge for the signal processing community for current DL models and future FMs by identifying and jointly incorporating three crucial aspects into an evaluation framework: performance uncertainty, learning efficiency, and overlap between training and test datasets. The first evaluation aspect, performance uncertainty (see Bouthillier et al., 2021), measures the variability in the model's performance arising from two sources of variation: the stochastic training process and the random training data sample. Characterizing the variation of a model's performance helps practitioners qualify expectations and guards against a model developer erroneously claiming a technological advancement. The second

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Samuel Myren · Nidhi Parikh · Rosalyn Rael · Garrison Flynn · Emily Casleton

Los Alamos National Laboratory, Los Alamos, NM, 87545

e-mail: [samuelymyren@gmail.com](mailto:samuelymyren@gmail.com); [nidhip@lanl.gov](mailto:nidhip@lanl.gov); [rrael@lanl.gov](mailto:rrael@lanl.gov); [garrison@lanl.gov](mailto:garrison@lanl.gov); [ecasleyton@lanl.gov](mailto:ecasleyton@lanl.gov)

Dave Higdon

Department of Statistics, Virginia Tech, Blacksburg VA, 24061

e-mail: [dhigdon@vt.edu](mailto:dhigdon@vt.edu)

evaluation aspect, learning efficiency (for reference see Hlynsson et al., 2019), measures the model’s capability to perform under varying sizes of training data. Learning efficiency is fundamental to FM evaluation because a user needs to understand the cost to benefit ratio of obtaining more labeled data during the fine-tuning stage. It also is imperative to current DL methods to determine which model can do more with less. The third evaluation aspect is the effect on model performance resulting from overlap in the training and test data (e.g., Elangovan et al., 2021), which directly targets the FM’s pre-training on big data phase where the test dataset may overlap or be partially/fully contained in the pre-training dataset.

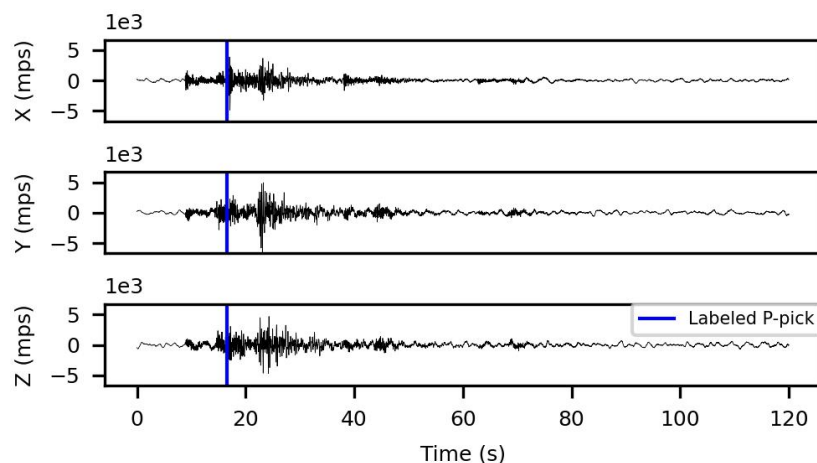
As of conducting this research, easily accessible foundation models were unavailable for the SHM community, so we demonstrate our evaluation framework by employing traditional DL methods borrowed from the seismic community. We focus on methods of interest to the SHM community by using a model trained to identify and temporally locate events in ground vibration data. We apply our framework to compare four competing models under various training methods including pre-training, fine-tuning, and trained-from-scratch. To address the three crucial evaluation aspects, we employ rigorous, thoughtful partitioning of the training, validation, and test data splits and we utilize random data selection and deep ensembles. We then compare the models with uncertainty across a variety of performance metrics and discuss the results.

## Background

To demonstrate the evaluation framework, we borrow a deep learning algorithm from the seismic community, PhaseNet (Zhu & Beroza, 2018). PhaseNet is a U-Net type architecture (Ronneberger et al., 2015) designed to identify characteristic signatures in tri-axial ground acceleration data (a waveform) that propagates through the Earth from an event’s (i.e. earthquake or blast) hypocenter to the station housing the seismometer. Prominent signatures include the primary (P) compressional wave and the secondary (S) longitudinal wave. The process of phase picking involves manually inspecting waveforms and identifying the time of arrival of the P/S-waves. Identification of these phases at various stations allows seismologists to locate and characterize an event. For demonstration purposes, we focus exclusively on identifying P-waves, called P-picking.

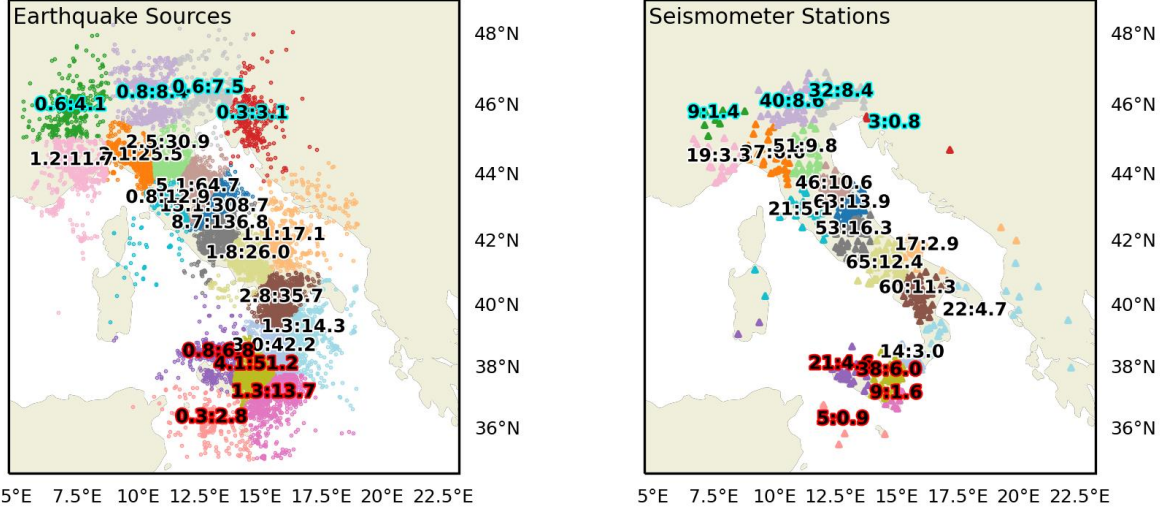
## Data

In this work, we use two publicly available, noncontinuous, labeled datasets. The first, the STanford EArthquake Dataset, STEAD (Mousavi et al., 2019), comprises  $\sim 1.3$  million waveforms including earthquakes and noise from sources and stations distributed across the globe. The second, the Italian seismic dataset for machine learning, INSTANCE (Michellini et al., 2021), also comprises  $\sim 1.3$  million earthquake and noise waveforms, but are regionally constrained near Italy. To measure the effect of overlap in training and test data, we spin off a third dataset we call STEAD Masked, which comprises all of STEAD data but removes the sources and stations that overlap with the Italian region. An example of a waveform featuring a labeled P-wave is shown in Fig. 1.



**Fig. 1** Example of a triaxial waveform of seismic data from the INSTANCE dataset with labeled P-pick (blue line).

To measure the three performance aspects, we carefully construct data splits using a clustering approach. We leave STEAD and STEAD Masked as they are because we primarily use them for pre-training models. For INSTANCE, since we aim to measure learning efficiency which tests model performance at various data amounts, we cluster the data using a K-means clustering algorithm based on source and station latitude and longitude (see Fig. 2). We employ 20 clusters as this provided a balance between cluster size and geographical division to help limit data leakage between clusters to better measure data sampling uncertainty and learning efficiency.



**Fig. 2** INSTANCE sources (left, circles) and stations (right, triangles) are clustered using 20 clusters based on geographical locations. For the left panel (sources), the text shows the number of sources in thousands followed by the number of waveforms in thousands. For the right panel (stations), the text shows the number of stations and the number of noise waveforms in thousands. The text highlighting indicates the group: cyan is north test group, white is training/validation group, and red is south test group.

With the INSTANCE clusters, we create the training, validation, and test splits to accomplish our evaluation goals. We use the 4 northern most clusters and 4 southernmost clusters to comprise the test set, balancing the data within each region approximately equally through random source sampling resulting in about 50,000 earthquake waveforms and 5,000 noise waveforms. The remaining 12 central clusters form the training and validation pool of data. To make the validation set, we randomly choose 159 sources and 276 noise waveforms from each cluster, totaling 27,169 earthquake waveforms and 3,312 noise waveforms to form the validation set. The remaining data from the 12 central clusters comprises the training pool. When training models, the amount of training data is varied and randomly chosen from the quantity of clusters assigned. In general, we aim to keep an 80-20 training to validation ratio, so each training cluster comprises 636 sources (which on average corresponds to 9,705 waveforms) and 1,106 noise waveforms.

## Methods

We apply the PhaseNet architecture (Zhu & Beroza, 2018) for all models which has been well researched for the phase picking domain. PhaseNet is a U-Net type architecture (Ronneberger et al., 2015) with identical input and output sizes of  $3 \times 3001$ . That is, the model accepts 30s tri-axial vibration data sampled at 100Hz and produces outputs of the same size. Thus, we train the model to predict the probability of 3 features as a function of time: one vector for predicting P-waves, one for S-waves, and one for noise. For demonstration, we discard the S-wave and noise predictions for all subsequent analysis.

To exhibit the versatility of the evaluation framework and to measure the effect of overlapping training and test data, we construct 7 competing models (see Table 1) for comparison using various training schemes. The first two models act as a baseline and are trained and validated on STEAD and STEAD Masked but tested on INSTANCE. As such, we refer to these models as out-of-distribution (OOD) models (OOD and OOD Masked, respectively). The next 4 models utilize transfer learning (TL) and exemplify a situation where we have a pre-trained model available and some data available for further training. We refer to these models as TL Free, TL Free Masked, TL Frozen, and TL Frozen Masked. The Masked

label indicates if the model is initialized using weights from the STEAD or STEAD Masked OOD models, and the Free versus Frozen label indicates whether half of the model’s weights have been fixed to mimic a fine-tuning approach common for foundation models. When freezing weights, we fix the first half of the model: all down sampling layers at their initialized values. The final model, Standard, represents the standard approach to deep learning, where the weights are randomly initialized, and the model is trained using INSTANCE data. All models are tested on the INSTANCE test regions.

**Table 1** Details of the 7 competing models are displayed. Each row represents a model, and the columns designate the model name, how the model was initialized (pre-trained or randomly initiated), whether the weights are free or fixed, the training and validation datasets, and the quantity of cluster levels that each model was trained under.

| Model Name       | Initialization | Weights     | Training Set | Validation Set | Quantity of Clusters |
|------------------|----------------|-------------|--------------|----------------|----------------------|
| OOD              | Random         | Free        | STEAD        | STEAD          | NA                   |
| OOD Masked       | Random         | Free        | STEAD Masked | STEAD Masked   | NA                   |
| Standard         | Random         | Free        | INSTANCE     | INSTANCE       | {1,2,6,9,12}         |
| TL Free          | OOD            | Free        | INSTANCE     | INSTANCE       | {1,2,6,9,12}         |
| TL Free Masked   | OOD Masked     | Free        | INSTANCE     | INSTANCE       | {1,2,6,9,12}         |
| TL Frozen        | OOD            | Half Frozen | INSTANCE     | INSTANCE       | {1,2,6,9,12}         |
| TL Frozen Masked | OOD Masked     | Half Frozen | INSTANCE     | INSTANCE       | {1,2,6,9,12}         |

To evaluate learning efficiency, we train each model under varying quantities of training clusters with levels 1, 3, 6, 9 and 12 clusters. We measure performance uncertainty arising from data sampling by training each model 12 times under each level, randomly choosing the cluster set that comprises the quantity of clusters each time. Then, to quantify training uncertainty, we hold the data fixed and utilize adaptations of deep ensembles (Ganaie et al., 2022; Huang et al., 2017) to retrain each model under 4 different weight initializations while keeping the training data fixed. Note that the two OOD models are exempt from this training scheme, as their training data remains fixed. However, to supply multiple initialization points for the other models, we retrain the two OOD models using deep ensembles 48 times each. For ease of discussion, we refer to a single “model instance” as the unique combination of model, quantity of clusters, cluster set, and initialization. For example, the TL Free model trained with 3 clusters comprising the randomly chosen cluster set of green, blue, and yellow (in reference Figure 1) and the 3rd initialization represents a single model instance. Thus, we obtain 1,200 model instances from the 5 non-OOD models and 96 model instances from the 2 OOD models, totaling 1,296 model instances.

## Evaluation

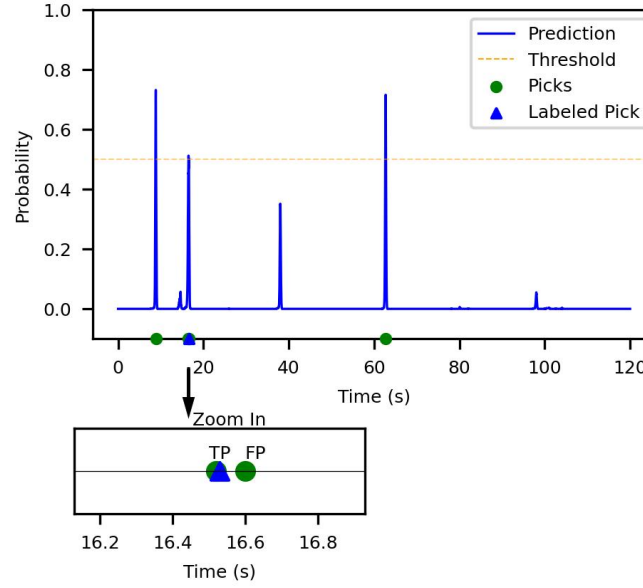
For this work, we focus on metrics that summarize the model performance in terms of identifying and localizing the P-wave within each earthquake waveform. The model produces a prediction of the probability of a P-wave for each time point in the waveform. We implement a user-defined threshold (see Fig. 3) where if the prediction exceeds the threshold, we obtain a predicted pick. We then classify the picks for each earthquake waveform in the test dataset. We obtain a true positive if the predicted P-pick is within 0.3s of the labeled P-pick, false positives for picks extra picks and those outside of 0.3s of the labeled P-pick, and false negatives if no predictions are made. We do not collect true negatives as they are relatively noninformative for this domain. We designate the total number of true positives, false positives, and false negatives as TP, FP, and FN and compute metrics including  $Precision = \frac{TP}{TP+FP}$ ,  $Recall = \frac{TP}{TP+FN}$ , and  $F1 = \frac{2TP}{2TP+FN+FP}$ .

With metrics of interest in hand, we present a statistical model to condense the information from the 1,296 model instances into a digestible form for comparison. Given a metric that summarizes the performance, designated  $y_{madi}$ , where  $m$  indexes one of the 7 models,  $a$  indexes the quantity of clusters,  $d$  indexes the cluster set, and  $i$  indexes each of the 4 initializations, we present the statistical model

$$y_{madi} = \gamma + \mu_m + \alpha_a + \theta_{ma} + \epsilon_{mad}^{data} + \epsilon_{madi}^{train}.$$

Since the OOD models are not trained over varying quantities of clusters and cluster sets, the statistical model simplifies by dropping the indices  $a$  and  $d$  and associated terms. In this model,  $\gamma$  is the grand mean,  $\mu_m$  is the effect of the modeling approach,  $\alpha_a$  is the effect of the quantity of clusters (i.e. data amount), and  $\theta_{ma}$  is the interaction term. Furthermore, we assume  $\epsilon_{mad}^{data} \sim N(0, \sigma_{ma}^{2data})$  and  $\epsilon_{madi}^{train} \sim N(0, \sigma_{ma}^{2train})$  where the variance arising from the data and the training are separable and independent terms based on the training design.

Through this model, we estimate the effects of each model and its relationship with the quantity of clusters to evaluate the learning efficiency of each model. Furthermore, we can exploit the normal assumption to obtain confidence intervals



**Fig. 3** An example of the model’s prediction is shown (blue) for a single waveform. The labeled P-pick (blue triangle) is shown along with other predicted picks made by the model (green circles). This prediction results in 1 true positive and 3 false positives. The inset plot on the bottom zooms in to the region very near the true positive to show how multiple picks are made.

around the effects. Finally, we can estimate each variance term and form confidence intervals to determine which model exhibits minimal variability.

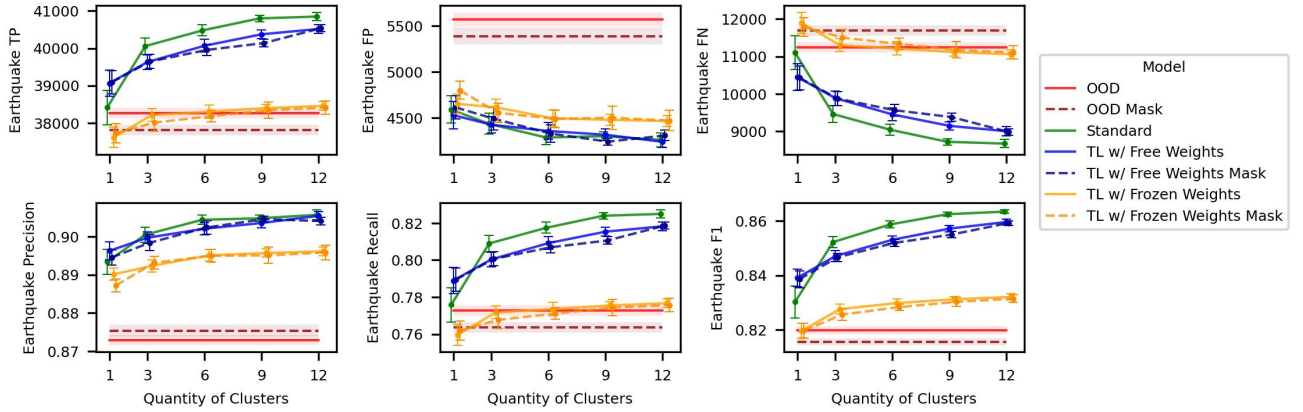
## Results

With the 1,296 trained model instances and metrics calculated, we cast the information into the statistical model to obtain mean performance and 90% confidence intervals for each metric. The results are summarized in Fig. 4. For the simpler metrics (total TP, FP, and FN), we see the largest gains across all models occurring between 1 and 3 quantities of clusters and that increasing the quantity of clusters beyond 3 results in diminishing improvements. In comparing the 7 models directly, we identify the general trend that the Standard (green) and two TL w/ Free weights (blue) outperform the other 4 models. However, if we were to rank the models based on performance, our rankings would change depending on the quantities of clusters as exhibited by non-parallel lines. For example, the Standard (green) model does worse than the TL w/ Free Weights (blue) models at 1 cluster, but quickly outperforms as more data is added. Similarly, the 2 TL w/ Frozen Weights (orange) models perform even worse than the 2 OOD models at 1 cluster but perform better as more data is added.

This last result, that the TL w/Frozen Weights (orange) perform worse for small data is certainly a surprising result. We may expect that by freezing half the weights and performing some re-training, we see some performance improvement as the weights are fine-tuned from their pre-trained states. However, this hypothesis is ill supported in the TP panel alone. Fortunately, the other metrics provide some context. We see that the orange models also have few FPs. This indicates that the orange models are predicting more conservatively in general, which explains the results in the precision, recall, and F1 panels. We see that precision for the orange models resides in between the OOD models and the rest of the models. This makes sense because the TL w/ Frozen Weights are effectively a hybrid among all the modeling approaches. These results convey the importance of continuing to evaluate current DL/AI models and future FMs against simpler training approaches and that model rankings can change depending on the amount of data being employed.

To determine the effect of overlap between training and testing data, we compare the solid lines (non-Masked) to their dashed lines (Masked) models. We see that the non-Masked models typically outperform their Masked counterparts, although the effect is small and inconsistent. The exceptions to this trend are the OOD models (red) for FP and recall, where the masking appears to improve performance. This case is likely explained by how the masked models predict with lower confidence in general and thereby results in fewer FPs. Thus, these results are as expected: removing training data that



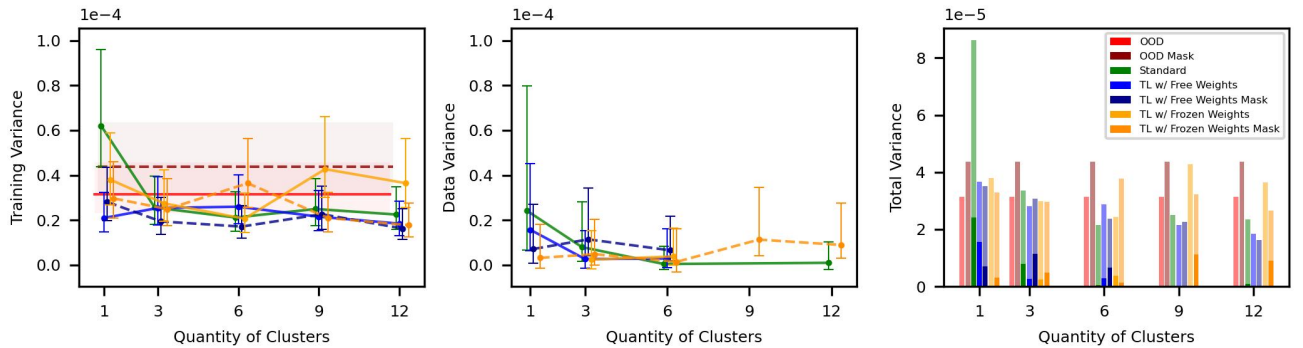


**Fig. 4** Results of the model performance for each of the 7 competing models across different quantities of clusters. Each panel is a different metric, including total TP, FP, and FN, and precision, recall, and F1. The OOD models (red and dark red) are constant because they are exempt from the training design. Ribbons and vertical bars represent 90% confidence intervals.

overlaps with the test data reduces model performance. This is an important finding for evaluation FMs because it tells us that we must hold out full datasets from the pretraining phase of all the FMs to obtain fair evaluation.

The importance of having error bars is worth emphasizing in Fig. 4. Consider if you were to train each of the 7 models only once under 1 cluster. Since each model instance that you train would come from the corresponding model's distribution of performance, you may mis-rank the models as exhibited by overlapping error bars. Thus, equipped with the uncertainty information, a practitioner when selecting the model can calibrate their expectations for a given model's performance.

In considering the performance error bars and ribbons in Fig. 4, we can decompose these intervals through estimating the two sources of uncertainty arising from training and data. These estimates for precision, along with 90% confidence intervals are shown in Fig. 5, however the uncertainty can be analyzed in this way for any metric. In the training variance, we see a generally flat trend as the quantity of clusters is increased. However, this trend is broken for the Standard (green) model where we see markedly higher training variability that decreases toward the trend as we move from 1 to 3 clusters. This tells us that for small data, we may avoid using the Standard model due to highly varying performance.



**Fig. 5** Variance estimates are shown for the metric precision for each model as a function of quantities of clusters. Training variance (left panel) and data variance (middle panel) are shown along with ribbons or error bars representing 90% confidence intervals. The right panel shows the total variability as a stacked bar chart showing the relative contributions of the training variance (transparent) and data variance (opaque) to the total variance.

For the data variance (middle panel in Fig. 5), we see decreasing trends as more data is added. This is expected, because as we add more data, we better approximate the full dataset, thereby reducing variability. In addition, as exhibited by the right panel where the total variance is shown along with relative contributions of each source, the data variance contributes minimally relative to the training variance. Thus, in the middle panel, some of the curves abruptly stop at 6 clusters. This is because, as an artifact of the statistical framework, we obtain negative values for the data variance. Since negative variance is

theoretically impossible, we choose to remove that information from the plot, but it essentially means that the data variance is extremely small relative to the training variance. Overall, understanding which aspect in the modeling process is contributing to variance allows developers to design models more efficiently and allows practitioners to choose models that best fit their needs.

## Conclusion

In this work, we demonstrate a robust statistical framework that jointly incorporates three crucial evaluation aspects for current AI/DL models and future FMs, including performance uncertainty, learning efficiency, and overlap between training and test data. We demonstrate our evaluation framework using existing DL models from the seismic phase picking community by comparing 7 competing models under various training schemes, including a standard, out-of-distribution, pre-training, and fine-tuning model deployments. We design careful data splits and train each model under varying training sizes with multiple initializations to obtain segregated estimates of uncertainty arising from training process and data sampling. We summarize model results by computing metrics of interest and cast the information into a statistical model to summarize the results. We exemplify the importance of including uncertainty in an evaluation framework and how the model can interact strongly with the data amount. Furthermore, we find a non-negligible effect of overlap between training and test data. In general, these results show the need for such an evaluation framework to fairly evaluate models under a variety of contexts and to inform practitioners such that they can choose the best model for a given set of constraints.

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## Chapter 12

# Uncertainty Quantification in Railway Bridge Dynamics via on-board Accelerations Collected from Traversing Trains

Charikleia Stoura, Szymon Grés, and Eleni Chatzi

**Abstract** Structural Health Monitoring (SHM) offers a pragmatic approach to inform on bridge condition and performance given the exposure of these structures to increasing loads, adverse environments, and ageing. Traditional SHM methods, which rely on sensors installed directly on bridge structures, face limitations such as high costs and insufficient infrastructure coverage. To address these issues, mobile sensing approaches have emerged in recent years. These methods utilize sensors mounted on vehicles traversing bridge elements, collecting data as they pass. In railway systems, this can involve dedicated diagnostic rail vehicles equipped with sensors or even regular in-service trains fitted with low-cost accelerometers. The data gathered by these vehicles enable the use of indirect monitoring approaches for identifying the dynamic properties of bridges, which can then be used to detect potential damage or deterioration. One significant challenge with indirect monitoring is ensuring that the identified modal properties can be attributed to the monitored structure, as opposed to the vehicle and other interaction phenomena, especially when no prior information about the expected properties is available. This study introduces an on-board monitoring approach that estimates the modal properties of railway bridges using acceleration data collected from passing trains, while simultaneously measuring the uncertainty of the identified modal parameters. The acceleration data is processed through a model-based methodology employing Bayesian inference to estimate the states and input (contact force) of the vehicle. Subsequently, a subspace identification approach, combined with a Delta method-based uncertainty quantification methodology, is used to estimate the modal frequencies of the traversed bridge and the associated uncertainty. The proposed methodology is demonstrated on a simple vehicle-bridge system using simulated acceleration data. Long-term monitoring of these frequencies can serve as an indicator of damage, facilitating timely maintenance and ensuring the safety and reliability of rail transportation.

**Keywords** Structural health monitoring · On-board monitoring · Subspace identification · Delta method · Uncertainty quantification

## Introduction

Bridges constitute a fundamental component of railway infrastructure, especially in the case of high-speed rail (HSR) systems. In HSR networks, bridges may constitute more than 50% of an HSR line's total length [1]. As rail networks expand, and with many bridges in older systems nearing the end of their operational lifespan, ensuring the structural integrity of bridges is critical for avoiding accidents or disruptions. Structural Health Monitoring (SHM) plays a key role in this respect, allowing for early detection of damage and hence facilitating maintenance planning [2]. Traditional SHM methods, such as deployment of fixed sensors on the structure or conducting visual inspections, are effective but costly and labor-intensive, especially for extensive rail networks [3]. To address this challenge, mobile sensing approaches, also known as indirect SHM, have been developed [4]. Such approaches rely on the vehicle-bridge interaction (VBI) phenomenon, i.e., on the coupled dynamics of vehicles and bridges when a vehicle traverses a bridge [5]. In these methods, vibrations recorded from vehicles

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Charikleia Stoura · Eleni Chatzi

Institute of Structural Engineering, Department of Civil, Environmental and Geomatic Engineering, ETH Zürich, Zürich, 8093, Switzerland

e-mail: [charikleia.stoura@ibk.baug.ethz.ch](mailto:charikleia.stoura@ibk.baug.ethz.ch); [chatzi@ibk.baug.ethz.ch](mailto:chatzi@ibk.baug.ethz.ch)

Szymon Grés

Department of Electronic Systems, Aalborg University, 9220 Aalborg, Denmark

e-mail: [sgres@es.aau.dk](mailto:sgres@es.aau.dk)

crossing the bridge can provide valuable information on the bridge's structural state. This paper focuses on improving the long-term monitoring of railway bridges using advanced SHM methods.

## Background

One key problem in indirect SHM of bridges is extracting the modal properties of the bridge, e.g., eigenfrequencies, from vehicle data in an accurate and reliable manner. Main challenges to this end are the interference of the recording vehicle frequencies and the roughness profile of the pavement (for highway bridges) or the tracks (for rail bridges). Subspace Identification (SID) algorithms have been widely used to identify frequencies, damping ratios, and mode shapes of bridges by modeling the system as a linear, time-invariant state-space representation and, at the same time, liberating from the presence of the vehicle frequencies in the identification task [6, 7, 8]. However, on-board monitoring relies on a recording vehicle moving on the sustaining bridge, which renders the state-space system of the VBI system time-variant. One approach to convert such a system into a time-invariant one is to assume slow variation of the dynamics of the vehicle with respect to the dynamics of the bridge by limiting the traversing speed of the vehicle [6]. Based on this assumption, Jin et al. [6] estimated frequencies of highway bridges based on an output-only short-time stochastic subspace identification (SSI) technique. Nevertheless, such an approach can not be applied to railway bridges, especially in the case of monitoring based on in-service trains, where moving speeds are typically much higher. To overcome this limitation, Jin et al. [8] approximated the first mode of the bridge as a linear or sinusoidal function, thus estimating the location of the moving vehicle on the bridge, and applied an input-output SID algorithm.

A common element of these SID approaches is that they assume known states of the vehicle at all degrees of freedom (DOFs) as well as known contact force. In practice, measuring such portions is difficult, if not impossible. Therefore, we proposed a methodology to estimate the modal properties of bridges based solely on the availability of measured accelerations from a recording railroad vehicle [9]. In doing so, a Bayesian inference approach was applied, namely a Dual Kalman Filter (DKF) [10], to first estimate the states of the vehicle and the exerted contact force [11], and then an input-output Multivariable Output Error State Space (MOESP) SID algorithm is used to estimate the frequencies of the bridge [8].

However, a significant issue with SID approaches is the uncertainty in the identified modal parameters, particularly when distinguishing actual bridge frequencies from those caused by noise or roughness. To address this, uncertainty quantification becomes essential. Greš et al. [12] introduced a first-order Delta method that propagates uncertainty in the data covariance onto the estimated modal parameters, allowing for a statistical evaluation of system parameters and providing confidence in the identified frequencies, damping ratios, and mode shapes.

This study presents an approach to estimate bridge modal properties from acceleration data of passing trains with a concurrent evaluation of the uncertainty in the identified modal parameters. Similar to our previous studies, we first estimate the states and input to the vehicle based on a Bayesian inference framework [11]. Then, we set up a subspace identification formulation for the bridge that eliminates the effect of rail roughness and assumes a known mode shape for the bridge in order to render the problem time-invariant [9]. Lastly, different from existing approaches, the subspace identification scheme concurrently performs an uncertainty quantification, ensuring that the identified bridge frequencies are not only accurately inferred, but also belong to the examined system [12].

## Analysis

The methodology is developed on a generic three-dimensional railroad vehicle-bridge system fig. 1 that assumes a known vehicle model. The dynamics of the vehicle are captured using a second-order equation of motion (EOM):

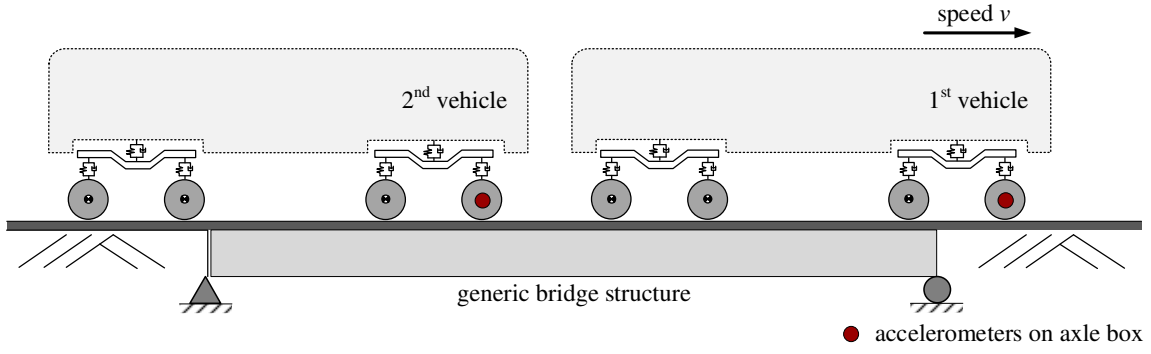
$$\mathbf{m}^v \ddot{\mathbf{u}}^v(t) + \mathbf{c}^v \dot{\mathbf{u}}^v(t) + \mathbf{k}^v \mathbf{u}^v(t) = \mathbf{W}^v \boldsymbol{\lambda}(t) \quad (1)$$

where  $\mathbf{u}^v(t)$  is the response vector of the vehicle, and  $\mathbf{m}^v$ , and  $\mathbf{c}^v$  and  $\mathbf{k}^v$  represent the damping and stiffness matrices, respectively.  $\boldsymbol{\lambda}(t)$  is the contact force vector between the vehicle and the underlying bridge system, and  $\mathbf{W}^v$  is the contact direction matrix connecting the DOFs of the vehicle to the contact force elements.

In turn, the bridge dynamics are described via an EOM of the form:

$$\mathbf{m}^b \ddot{\mathbf{u}}^b(t) + \mathbf{c}^b \dot{\mathbf{u}}^b(t) + \mathbf{k}^b \mathbf{u}^b(t) = -\mathbf{W}^b(x) \boldsymbol{\lambda}(t) + \mathbf{F}^b \quad (2)$$

where  $\mathbf{u}^b(t)$  is the response vector of the bridge, and  $\mathbf{m}^b$ ,  $\mathbf{c}^b$ , and  $\mathbf{k}^b$  are, respectively, the mass, damping, and stiffness matrices.  $\mathbf{W}^b(x)$  is the contact direction matrix of the bridge, which changes with time  $t$ . Finally, the expression of the



**Fig. 1** Two railroad vehicles traversing a generic bridge with accelerometers mounted on the axle boxes of their first wheelsets (red circles).

contact force follows an elastic contact model as:

$$\lambda = k \left[ (\mathbf{W}^b(x))^T \mathbf{u}^b(t) - (\mathbf{W}^v)^T \mathbf{u}^v(t) + \mathbf{r}(x) \right] \quad (3)$$

where  $k$  is a contact stiffness parameter that multiplies the relative displacement between the wheels of the vehicle and the sustaining bridge and  $\mathbf{r}(x)$  is the vector containing the roughness at each contact point, assuming that the vehicle traverses the bridge running on tracks with a rough surface.

After modeling the dynamics of the coupled system, the first step is to estimate the unknown contact force between the railroad vehicle and the bridge. Based on the assumption that the railroad vehicle dynamics are known while accelerations are measured at the axle boxes of the vehicle, the states (displacement, velocity, and acceleration) of the system and the contact forces acting on it can be estimated. To this end, this study employs a DKF [10] for simultaneous state and input estimation. In this case, the vehicle is written in a state-space form as:

$$\dot{\mathbf{x}}^v(t) = \mathbf{A}^v \mathbf{x}^v(t) + \mathbf{B}^v \lambda(t) \quad (4)$$

where  $\mathbf{x}^v(t)$  is the state vector of the vehicle containing displacements and velocities, and  $\mathbf{A}^v$  and  $\mathbf{B}^v$  are the state and input matrices containing the known vehicle properties. Accordingly, the output equation, which links the measured accelerations  $\mathbf{z}(t)$  to the vehicle's states and contact forces, is:

$$\mathbf{z}(t) = \mathbf{C}^v \mathbf{x}^v(t) + \mathbf{D}^v \lambda(t) \quad (5)$$

where  $\mathbf{C}^v$  and  $\mathbf{D}^v$  are the output and feedforward matrices, respectively, also estimated via the known vehicle properties. Finally, the system of equations is discretized, and a random walk model is assumed for the evolution of the unknown contact force.

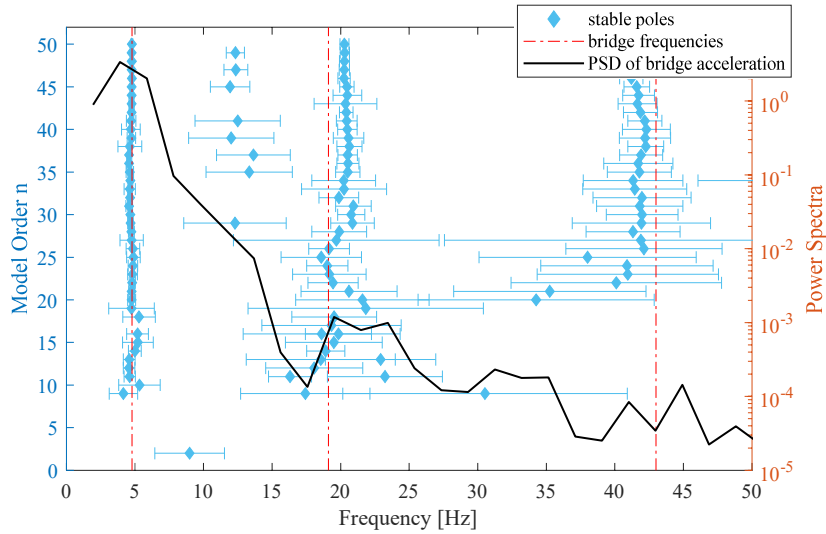
Upon estimation of the railroad vehicle states and inputs, the bridge modal properties are inferred following a SID approach. To this end, the bridge EOM is written in a state-space form:

$$\dot{\mathbf{x}}^b(t) = \mathbf{A}^b \mathbf{x}^b(t) + \mathbf{B}^b \mathbf{W}^b(x) \lambda(t) \quad (6)$$

where  $\mathbf{x}^b(t)$  is the state vector of the bridge, and  $\mathbf{A}^b$  and  $\mathbf{B}^b$  are the state and input matrices. In eq. (6),  $\lambda(t)$  is known as it was estimated from Eqs. eqs. (4) and (5) via the DKF. To apply a SID algorithm on eq. (6), an output equation is also necessary, which is formulated based on the EOM of the vehicle eq. (1). However, to bring the vehicle EOM into a proper output equation that also includes the bridge response, the expression of the contact force (eq. (3)) is substituted into eq. (3). In this case, the bridge response  $\mathbf{u}^b(t)$  and roughness profile  $\mathbf{r}(x)$  appear in the output equation. The roughness profile cannot be directly identified and is, thus, eliminated from the output equation by assuming measurements by two consecutive vehicles crossing the same roughness profile, as demonstrated in fig. 1 and considering the residual response of the two vehicles. The residual response is:

$$\Delta \mathbf{s}(t) = \mathbf{W}^v k (\mathbf{W}^b(x))^T (\mathbf{u}_2^b(t) - \mathbf{u}_1^b(t)) \quad (7)$$

where  $\Delta \mathbf{s}(t)$  contains the states of the first and second vehicles that were estimated via the DKF. Note that in eq. (7) no roughness profile appears. Eventually, the state-space EOM of the bridge (eq. (6)) is also written as the difference of the



**Fig. 2** Example of stabilization diagram from the residual response of two single-degree-of-freedom vehicles traversing a simply supported bridge with a simultaneous estimation of the uncertainty of the estimated bridge frequencies.

bridge response when the two vehicles cross the bridge as:

$$\dot{\mathbf{x}}_2^b(t) - \dot{\mathbf{x}}_1^b(t) = \mathbf{A}^b (\mathbf{x}_2^b(t) - \mathbf{x}_1^b(t)) + \mathbf{B}^b \mathbf{W}^b(x) (\lambda_2(t) - \lambda_1(t)) \quad (8)$$

and the output equation (eq. (7)) in state-space assumes the form:

$$\Delta \mathbf{s}(t) = \mathbf{H}(x) (\mathbf{x}_2^b(t) - \mathbf{x}_1^b(t)) \quad (9)$$

where  $\mathbf{H}(x)$  includes quantities associated with  $\mathbf{W}^b(x)$  which is here considered to be known, assuming a sinusoidal shape mode for the bridge.

The solution of eqs. (8) and (9) follows an input-output covariance-driven SID algorithm that uses an orthogonal projection of the future output Hankel matrix onto the future input Hankel matrix, which is then right-multiplied with the transpose of the part output data Hankel matrix. The Hankel matrix is then decomposed using Singular Value Decomposition (SVD) to extract the systems's modal parameters. To estimate the uncertainty of the estimated modal parameters, this study adopts the Delta method, as demonstrated by Greš et al. [12]. Specifically, it propagates the covariance of the data onto the modal parameter estimates and, hence, estimates the covariance of the estimated modal parameters. Eventually, the uncertainty of the modal properties of the system, e.g., modal frequencies, is estimated for the different orders assumed for the subspace identification, as demonstrated in fig. 2.

## Conclusion

This study proposes a methodology to perform on-board monitoring of rail bridges based on acceleration data of passing railroad vehicles with a simultaneous estimation of the uncertainty of the identified modal parameters. By employing a DKF for state-parameter estimation we only require measured accelerations from the recording vehicle, overcoming the limiting assumption of known vehicle displacements, velocities, and contact forces for the performance of the identification task, which is a common assumption in previous studies. Subsequently, by incorporating an uncertainty quantification scheme within the modal identification task we enhance the reliability of subspace identification for modal parameter estimation. This methodology offers a cost- and time-efficient solution for indirect monitoring of rail infrastructure and paves the way for scalable online monitoring of rail assets.

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## Chapter 13

# Optimal Experiment Design for Large-Scale Inverse Problems with Enhanced Robustness to Model Uncertainties



Wilkins Aquino, Mark Chen, Drew Kouri, Kathryn Maupin, Joshua Mullins, Chandler Smith, Timothy Walsh, and Rebekah White

**Abstract** In structural dynamics, response data obtained from laboratory-controlled vibration tests or real-time monitoring are essential to inform uncertain model parameters such as structural loads, boundary conditions, or material properties. Often, response data are limited due to cost or physical constraints on the number of sensors that can be placed in an experiment. Optimal experimental design (OED) is a powerful tool for judiciously choosing what experiments will provide the most informative data, i.e. the data that reduces uncertainty in resulting parameter estimates, which is characterized by scalar-valued functions of the associated asymptotic covariance. However, such classical formulations of OED ignore the impact of uncertainties associated with auxiliary parameters – model parameters not inferred from experimental data but whose values impact the accuracy of model predictions. Therefore, this work considers more robust OED strategies that account for uncertainty in the auxiliary parameters when predicting an optimal experimental design. Specifically, we characterize uncertainty in the auxiliary parameters using probability distributions; the classical alphabetical criteria (e.g.  $I$ - or  $D$ -optimality) are then computed with respect to this distribution. To ensure a scalar-valued OED objective function, minimax and expectation-based approaches are used to minimize worst-case performance or maximize average performance, respectively. We consider two approaches for solving the optimization problem determining the best design: 1) utilizing a greedy algorithm for binary weights and 2) performing gradient-based optimization for fractional weights. To overcome the computational expense associated with the large-scale finite element models (FEM) structural dynamics code, we develop a reduced order model (ROM) to utilize within the robust OED strategies. Specifically, a reduced basis method (based on formal *a posteriori* error estimation concepts) is used to construct the ROM. Using an optimal control inverse problem within the context a ground-based vibration test, we demonstrate the effectiveness of robust OED in comparison to standard OED approaches. Furthermore, we demonstrate how computational intractability inherent to large-scale structural dynamics problems can be overcome through integration of robust OED strategies with reduced order modeling.

**Keywords** Robust Optimal Experimental Design · Large-Scale Inverse Problems · Experimental Design Under Uncertainty · Reduced Order Modeling

## Introduction

Across many engineering applications, accurately representing a physical system of interest of requires large-scale finite-element models (FEM), which can have on the order of millions of degrees of freedom. Furthermore, the parameters comprising such models are often uncertain or even unknown, requiring experimental data for calibration. Within the field of structural dynamics, inverse methods are employed to estimate unknown model parameters such as structural loads or material properties, using vibration response data. One challenge typical to inverse methods is the extraction of comprehensive information regarding the target unknown parameters from, what is often, limited experimental data; for example, laboratory-controlled vibration tests or real-time monitoring, can be costly, limiting the amount of feasible tests. Furthermore, within an

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Wilkins Aquino

Department of Mechanical Engineering and Materials Science, Duke University, Durham, NC, 27708

e-mail : [wa20@duke.edu](mailto:wa20@duke.edu)

Mark Chen · Drew Kouri · Kathryn Maupin · Joshua Mullins · Chandler Smith · Timothy Walsh · Rebekah White

Sandia National Laboratories, Albuquerque, NM, 87112

e-mail: [marchen@sandia.gov](mailto:marchen@sandia.gov); [dpkouri@sandia.gov](mailto:dpkouri@sandia.gov); [kmaupin@sandia.gov](mailto:kmaupin@sandia.gov); [jmullin@sandia.gov](mailto:jmullin@sandia.gov); [chasmit@sandia.gov](mailto:chasmit@sandia.gov);

[tfwalsh@sandia.gov](mailto:tfwalsh@sandia.gov); [rebwhit@sandia.gov](mailto:rebwhit@sandia.gov)

experiment, there are limitations to the number of sensors that can be placed on a physical system. With such limited data, it is imperative to perform experiments that yield the most information.

Optimal experimental design (OED) is a valuable tool for predicting *a priori* what experiments will be most informative given an experimental budget. An often employed, standard approach to OED, is to relate the informativeness of data to how well that data allows for the estimation (calibration) of the underlying model parameters; here  $M$ -estimation (e.g. least squares, maximum likelihood) [1] is used to estimate the model parameters while the corresponding asymptotic covariance provides a measure of uncertainty in the estimate [2]. As a result, scalar-valued functions of the asymptotic covariance are often used to formulate an OED objective function. Examples include the trace ( $A$ -optimal) and determinant ( $D$ -optimal) of the covariance. Within the context of  $M$ -estimation, robust design approaches have been considered. For example [3] formulates designs that are robust to misspecified response functions (given by additive model corrections) and unexpected heteroscedasticity. For nonlinear models, the optimality criterion depends upon the estimated parameters. Thus, uncertainty in the estimated parameters affects the design; approaches accounting for this uncertainty include [4, 5], which leverage minimax strategies, and [6], which leverages expectation-based strategies. Such formulations, however, do not account for uncertainties associated with auxiliary parameters, i.e. model parameters not inferred from experimental data but whose values impact the accuracy of structural dynamics model predictions.

Therefore, this work leverages robust OED strategies to account for uncertainty in the auxiliary parameters when predicting optimal experimental designs. Specifically, we characterize uncertainty in the auxiliary parameters using probability distributions. The OED objective is then computed with respect to this distribution using minimax and expectation-based approaches. The probabilistic representation of and subsequent marginalization over the auxiliary parameters follows from similar approaches in a Bayesian paradigm; see, e.g. [7]. However, here the OED criteria is formulated with frequentist estimates of the asymptotic covariance and risk measures beyond just the expected value are employed. We consider a structural dynamics model acted upon by unknown loads with material model uncertainty, where the experimental design is defined as the location at which sensors measuring vibration response are placed. For such sensor placement problems, the experimental design corresponds to weights of one or zero, indicating whether a sensor is placed or not, respectively; however, combinatorial complexity makes solving such optimization problems intractable. Thus, this work considers 1) utilizing a greedy approach, which selects an optimal sensor location one at a time [8] and 2) assigning probability masses (i.e. fractional weights) to the sensor locations, which allows for gradient-based optimization to be employed [9]. To overcome the computational expense associated with large-scale finite element models (FEM), we develop a reduced order model (ROM) to utilize within the robust OED strategies. Specifically, a reduced basis method (based on formal *a posteriori* error estimation concepts) is used to construct the ROM.

The main contributions of this work are as follows:

1. Demonstrate the effectiveness of robust OED in comparison to standard OED approaches for a structural dynamics force estimation inverse problem.
2. Provide a comparison between utilizing a greedy algorithm for binary weights and performing gradient-based optimization on weights representing probability masses.
3. Demonstrate how a reduced order model can enable computational tractability of robust OED strategies for structural dynamics applications.

The remainder of this paper is organized as follows: We begin by providing the relevant background for robust OED generally, highlighting how gradient-based and greedy approaches enable tractable optimization. Following this, we introduce the structural dynamics problem of interest, the wedding cake model, and provide the formulation for the ROM utilized within the OED framework. We then present the computational results, which include verification of the ROM accuracy, verifying the proposed greedy-based method, and comparing the gradient-based optimization strategy to the greedy method in terms of model prediction error. Lastly, final conclusions are drawn.

## Background and Methods

First, we present the general framework for robust optimal experimental design (OED). In this work's approach to robust OED, the aim is to determine what noisy observational data  $\mathbf{y}_\epsilon \in \mathcal{Y} \subset \mathbb{R}^{n_y}$  best inform a subset of parameters  $\boldsymbol{\theta} = [\theta_1, \dots, \theta_{n_p}] \in \Theta \subset \mathbb{R}^{n_p}$  of a mathematical model  $f(\boldsymbol{\theta}; \boldsymbol{\xi}; \mathbf{d}) : \Theta \times \Xi \rightarrow \mathcal{Y}$ , while accounting for the *irreducible* uncertainty in auxiliary parameters  $\boldsymbol{\xi} = [\xi_1, \dots, \xi_{n_a}] \in \Xi \subset \mathbb{R}^{n_a}$ , characterized by the probability distribution  $\pi(\boldsymbol{\xi})$ . The experimental design vector is denoted  $\mathbf{d} \in \mathcal{D} \subset \mathbb{R}^{n_d}$  and controls how data is collected; for example,  $\mathbf{d}$  may represent spatial locations at which response measurements are taken. We assume additive noise, and thus model the data as

$$\mathbf{y}_\epsilon = f(\boldsymbol{\theta}; \boldsymbol{\xi}; \mathbf{d}) + \boldsymbol{\epsilon}, \quad (1)$$



where  $\epsilon \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ , for an identity matrix  $\mathbf{I} \in \mathbb{R}^{n_y \times n_y}$ . Let the asymptotic covariance associated with the  $M$ -estimation of  $\theta$  be denoted  $C(\mathbf{d}) \in \mathbb{R}^{n_p \times n_p}$ . The OED criterion  $\Psi(\mathbf{d}, \xi)$  provides a scalar-valued representation of the covariance matrix; classical choices include  $D$ - and  $A$ -optimal criteria, which represent the determinant and trace of the covariance matrix, respectively. Then, the OED objective  $U : \mathcal{D} \rightarrow \mathbb{R}$  characterizing how “good” a given experimental design  $\mathbf{d}$  is can be formulated in a robust way as follows:

$$U(\mathbf{d}) = \mathcal{R}_\xi[\Psi(\mathbf{d}, \xi)], \quad (2)$$

where  $\mathcal{R}_\xi$  represents a risk measure that allows one to compute an “average” (with respect to  $\pi(\xi)$ ) objective. For example, letting  $\mathcal{R}_\xi := \mathbb{E}_\xi$  would result in the following expectation-based robust objective

$$U(\mathbf{d}) = \int_{\Xi} \Psi(\mathbf{d}, \xi) d\pi(\xi), \quad (3)$$

which we refer to as a risk-neutral approach. Alternatively, one could consider risk measures that penalize tail statistics, such as the average value-at-risk [10]. The optimal design  $\mathbf{d}^*$  is the one solving the following optimization problem

$$\mathbf{d}^* = \arg \min_{\mathbf{d} \in \mathcal{D}} U(\mathbf{d}). \quad (4)$$

In general, sensor placement problems utilizing binary weights result in (4) being combinatorially complex and thus intractable for expensive high-fidelity models and high-dimensional design spaces. Therefore this work considers two approaches to solving (4). The first is to consider a greedy optimization strategy, which aims to determine optimal sensor locations one (or more generally one batch) at a time. Here, the design is a vector of weights  $\mathbf{d} = [d_1, \dots, d_{n_d}]$ , where  $d_i \in \{0, 1\}$  for all  $i$ , and the corresponding optimization problem is given as

$$\mathbf{d}^* = \arg \min_{\mathbf{d} \in \{0,1\}} U(\mathbf{d}), \quad \sum_i d_i = N, \quad (5)$$

where  $N$  represents the sensor budget. The design weights  $d_i$  are determined sequentially, providing an approach that scales with the number of candidate sensor locations and requires  $\mathcal{O}(Nn_d)$  computations. Following [8], Algorithm 1 describes the greedy procedure. While not being guaranteed to find the optimal design, greedy algorithms have been shown to perform well in practice [11] and have theoretical justifications; see, e.g. [12, 13].

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**Algorithm 1:** Greedy sensor placement

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**Input:** Target number of sensors  $N$ .

**Output:** Design vector  $\mathbf{d}$

set  $\mathbf{d} = \mathbf{0}$ ,  $\mathcal{U} = \{1, \dots, n_d\}$ , and  $\mathcal{S} = \emptyset$ .

**for**  $k = 1 : n_d$  **do**

$i = \arg \min_{j \in \mathcal{U} \setminus \mathcal{S}} U(\mathbf{d} + \mathbf{e}_j)$

$\mathbf{e}_j$  is the  $j$ th coordinate vector in  $\mathbb{R}^{n_d}$

$\mathcal{S} = \mathcal{S} \cup \{i\}$

$\mathbf{d} = \mathbf{d} + \mathbf{e}_i$

**end**

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An alternative to utilizing a greedy algorithm is to instead allow the design vector to represent probability masses and solve the following optimization problem

$$\mathbf{d}^* = \arg \min_{\mathbf{d} \in [0,1]} U(\mathbf{d}), \quad \sum_i d_i = 1, \quad (6)$$

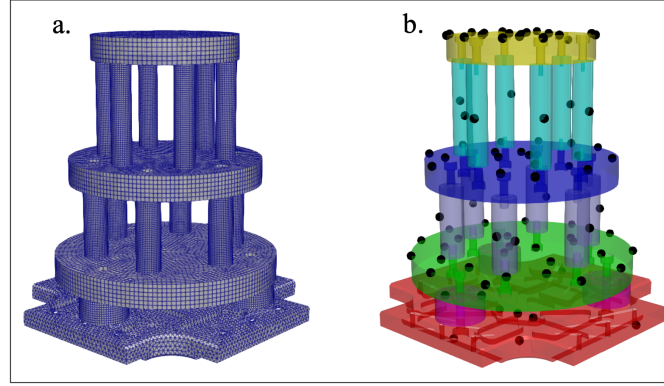
utilizing gradient-based optimization. Here, design vector weights reflect a frequency at which each sensor is utilized; thus, in practice, the larger the weight, the “better” the sensor location. Note that although this framing does not naturally enforce a given experimental budget  $N$ , for the problem considered herein, the number of candidate sensor locations is significantly larger than the number of inversion parameters, guaranteeing a sparse design [14]. Furthermore, heuristic approaches can be employed that select the  $N$  largest weights and utilize rounding and rescaling to ensure the interpretation of weights as probability masses.

## Model problem and reduced-order basis construction

Here, we present the so-called wedding cake model depicted in Figure 1a used to demonstrate the robust OED approach. We also present the reduced order model formulation that enables computationally efficient evaluation of the robust OED objective function. Vibration response is modeled mathematically according to the following linear parameter-to-observable map

$$f(\theta := s, \xi) := H(\xi)s, \quad (7)$$

where  $H$  is the frequency response function (FRF) matrix,  $s \in \mathbb{C}^{N_p}$  are the applied forces we wish to infer from experimental data, and  $\xi \in \Xi \subseteq \mathbb{R}^{n_a}$  are auxiliary parameters representing frequency as well as the density, elastic modulus, and Poisson's ratio of the columns on each floor, thus totaling ten auxiliary parameters.



**Fig. 1** a) Finite element model of the wedding cake and b) candidate sensor locations superimposed over the model. Uniaxial sensors can be oriented in one of three degrees of freedom.

Irreducible uncertainty in the auxiliary parameters is characterized by a multivariate uniform distribution. We assume a large variance in the uncertainty to encourage significant changes in the modal properties of the structure, in-turn eliciting complex dynamic response behavior.

The high-fidelity (HiFi) model solution is computed using Sandia's Sierra-SD FEM software to solve the finite element solution of the Helmholtz equations [15]. The frequency range of interest is 100Hz to 500Hz spaced at 8Hz intervals, thus totaling 50 frequency lines; on 60 processors, the runtime for one HiFi simulation using 50 frequencies is  $\sim 10$  minutes. Due to the computational costs of evaluating the HiFi model at every sample of the auxiliary parameters, a reduced order model (ROM) is needed to make subsequent experimental design computationally tractable.

We now provide an abbreviated overview of the adaptive reduced basis method of reduced order modeling in the context of approximating the FRF matrix. The formulations and algorithms presented herein are based on the work published in [16]. Consider the following equation for the finite element (FE) approximated Helmholtz problem describing the vibration response of a linear, time-invariant structure with  $n$  total degrees of freedom (DoFs),

$$\mathcal{M}(\hat{u}; \xi, \omega) \equiv A(\xi)\hat{u} - F = 0, \quad (8)$$

$\hat{u}(\xi) \in \mathcal{U}$  is the solution of the Helmholtz problem representing the vibration response,  $F$  is the forcing function,  $\omega$  is the circular frequency, and  $\mathcal{U} \subseteq \mathbb{C}^n$  represents the space of solutions. Note that  $A(\xi) \equiv K(\xi) + j\omega C(\xi) - \omega^2 M(\xi)$ , where  $K$ ,  $M$ , and  $C$  are the stiffness, mass, and damping matrices, respectively.

Our goal is to construct an efficient approximation (i.e. avoiding the solution to (8))  $\bar{u} : \Xi \rightarrow \mathcal{U}$  that is close to  $\hat{u}$  (in some sense). Consider a set of samples  $\Theta := \{\xi_i\}_{i=1}^k$ , referred to as atoms, drawn from the material parameter set  $\Xi$ . According to the reduced basis (RB) method [16], a finite dimensional subspace  $\mathcal{U}_k$  for the true solution space  $\mathcal{U}$ , is given as

$$\mathcal{U}_k \equiv \text{span}(\Phi_k := \{\hat{u}_1, \hat{u}_2, \dots, \hat{u}_k\}), \quad (9)$$

where  $\hat{u}_k := \hat{u}(\xi_k)$  is the solution to the Helmholtz problem given  $\xi_k$ , and  $\Phi_k \in \mathbb{C}^{n \times k}$ . We can now use (9) to define a reduced-basis approximation to the model solution as

$$\bar{u}_k(\xi) := \Phi_k a_k(\xi), \quad (10)$$

where  $a_k(\xi)$  solves the reduced problem

$$\Phi_k^T A(\xi) \Phi_k a_k = \Phi_k^T F. \quad (11)$$

The computational savings stem from the small number of basis vectors in  $\Phi_k$ , where the number of basis vectors  $k$  is much smaller than the dimension of the state space  $n$ .

In this work, the construction of the subspace  $\mathcal{U}_k$  is performed adaptively by sequentially selecting atoms based on the current surrogate solution  $\hat{u}$ . Given a current set of atoms, we select the next atom  $\hat{\xi}_{k+1}$  by identifying the atom corresponding to the largest expected ROM error over the parameter space. To avoid directly computing the ROM solution error, we estimate the error using a reliable *a posteriori* error estimate  $\epsilon_u(\xi)$ . Let the solution error be given by  $e(\xi) := \hat{u}(\xi) - \bar{u}(\xi)$ , and define the residual as  $r(\xi) := F(z) - H(\xi)\bar{u}(\xi)$ . We can show that (see [16])

$$\|e(\xi)\|_U \leq \frac{\|r(\xi)\|}{\gamma(\xi)} =: \epsilon_u(\xi), \quad (12)$$

where the inf-sup constant is given as  $\gamma(\xi) := \min_i |\lambda_i(\xi)|$ , with  $\lambda_i(\xi)$ ,  $i = 1, \dots, n$  being the eigenvalues of  $H(\xi)$ .

There are three major computational hurdles related to training the ROM: evaluating the inf-sup constant, reassembling the operator  $H$ , and training the ROM for a wide band of circular frequencies. In this work, to avoid solving a costly eigenvalue problem, we ignore the inf-sup constant and take the error estimator as simply  $\|r(\xi)\|_2$ . To address the large computational cost of assembling  $H$  for all background samples of  $\xi$ , we exploit the mathematical structure when the operator  $H$  is affine in the uncertain parameters; see [16] for details on the affine decomposition. Finally, to avoid computing a ROM for each frequency  $\omega$  of interest, we treat frequency as an uncertain parameter for the training phase of the ROM only (i.e.  $\omega \in \xi$ ).

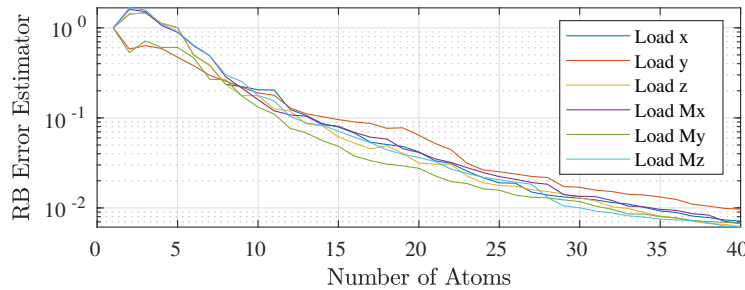
## Computational Results

### Reduced order model verification

First, we establish that the ROM for the wedding cake has adequate accuracy for use in robust OED. Utilizing the framework presented in the previous section, we train six independent ROMs using the reduced basis (RB) method, where each ROM corresponds to one column of the FRF matrix. Note that the ROM depends only on the load position and orientation, and scales linearly with the unknown load amplitude. Hence, we assume a unit load at the DoF corresponding to each unknown load when training each ROM.

In this example problem, the candidate acceleration measurement locations are shown as nodes in Figure 1b. Note that sensors can be oriented in one of three Cartesian directions, resulting in  $n_d = 267$  possible sensor DoFs (locations and orientations). Acceleration data is then used to estimate the unknown inference parameters ( $\theta$ ) given by six coincident point loads (three translational and three rotational) that excite the base.

Each ROM requires 40 HiFi model evaluations to achieve an acceptable degree of accuracy. Figure 2 presents the ROM error estimators (see (12)) as a function of atoms (i.e. HiFi model samples); here, we see adequate decay in error to establish the fidelity of the ROM during training. Noting that each HiFi solution is evaluated at one sample of the auxiliary parameters  $\xi$  (i.e. one sample of the material properties and a single frequency), a single ROM require approximately 100 minutes ( $40 \times 2.5$  minutes) to train. In total, the six ROMs require approximately 10 hours to train on 60 processors.



**Fig. 2** The reduced basis (RB) training error estimation as a function of atoms (i.e. samples of the HiFi model). Each legend item corresponds to a load.

For verification, we compare the solution to the Helmholtz problem using the trained ROM and the HiFi finite element model. Here, we assume the forces applied to the six DoFs at the model's base are also random variables described by a standard normal distribution to avoid skewing the verification results to one particular set of loads. The force amplitude is held constant across frequencies. The verification set used to compare the trained ROM and the HiFi model is comprised of 50 samples of the applied force distribution and auxiliary parameters. For each sample, the HiFi model and ROM are evaluated across a frequency range of 100 to 500 Hz at 8 Hz intervals. Note that the ROM and HiFi models are compared at the candidate sensor DoFs only.

Figure 3a presents the relative error between the ROM and HiFi solution for each sample of the verification set, where the relative error is taken as,

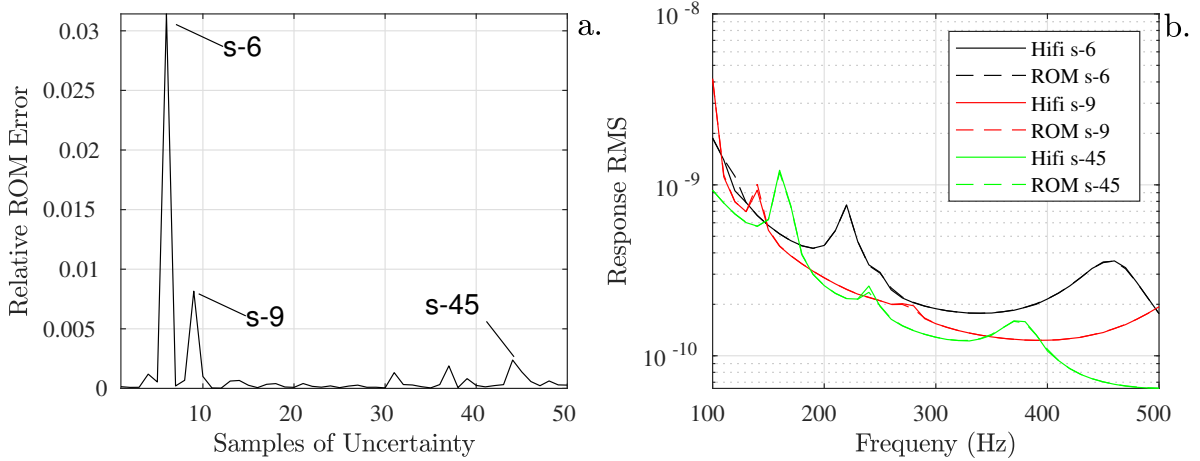
$$Relative\ Error = \frac{\|\hat{u}(\xi_k) - \bar{u}(\xi_k)\|_2^2}{\|\hat{u}(\xi_k)\|_2^2}, \quad (13)$$

where the  $l_2$ -norm is given by

$$\|x(\xi_k)\|_2 = \sqrt{\sum_{i=1}^{n_\omega} \sum_{j=1}^{n_d} |x_j(\xi_k, \omega_i)|^2}, \quad (14)$$

and where  $x_j(\xi_k, \omega_i)$  is the HiFi (or ROM) acceleration response evaluated at the  $j^{\text{th}}$  DoF for the  $i^{\text{th}}$  frequency corresponding to the random sample  $\xi_k$ . From 3a, for most random samples, the relative ROM error is low; three samples corresponding to peaks in uncertainty are noted as s-6, s-9, and s-45.

To understand how the relative error as a function of uncertainty samples propagates to response error, Figure 3b presents the root mean square (RMS) of the acceleration response over all DoFs as a function of frequency using the HiFi and ROM models. In Figure 3b, parameter samples corresponding to the three largest relative ROM errors (s-6, s-9, s-45) are used for clarity. From Figure 3b, we see that across significant shifts in resonances between the three model samples, the ROM corresponds well to the HiFi solution.



**Fig. 3** a) The relative error of the ROM with respect to the HiFi model as a function of the samples of uncertainty, where the samples corresponding to the three largest uncertainties are noted as s-6, s-9, and s-45. b) The RMS responses of the HiFi model and ROM corresponding to s-6, s-9, and s-45.

### Robust OED utilizing greedy optimization

With the verified accuracy, we can now evaluate robust OED approaches utilizing the ROM. Here, we focus on strategies utilizing the greedy approach presented in Algorithm 1. In this section, we consider the D-optimal expectation-based (i.e. risk-neutral) and minimax objective functions respectively given by

$$U(d) = \mathbb{E}_\xi [\Psi(d, \xi)] \approx \frac{1}{n_a} \sum_{i=1}^{n_a} \Psi(d, \xi_i), \quad \xi_i \sim \pi(\xi) \quad (15)$$

and,

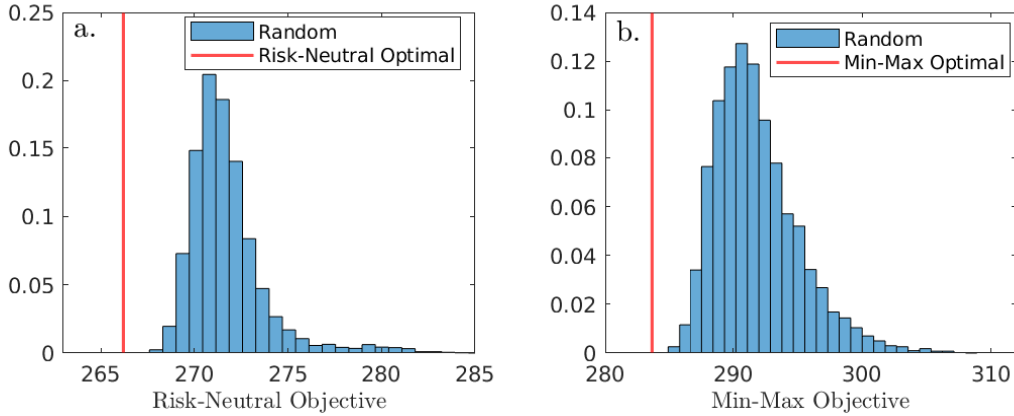
$$U(\mathbf{d}) = \max_{\xi \in \Xi} \Psi(\mathbf{d}, \xi) \approx \max_{i=1 \dots n_a} \Psi(\mathbf{d}, \xi_i), \quad \xi_i \sim \pi(\xi) \quad (16)$$

for,

$$\Psi(\mathbf{d}, \xi) = \log \det(C(\mathbf{d}, \xi)).$$

The robust experimental design is computed using 1,000 realizations of the material uncertainty and frequency to compute a corresponding 1,000 FRF matrix samples. Note that without the ROM, the total runtime for utilizing the HiFi model even with a parallelized greedy approach, is on the order of days. In contrast, utilizing the ROM requires on the order of minutes to compute an optimal design. To verify the performance of a given sensor configuration, we constructed a verification set using the same samples used to solve the robust OED problem. From this verification set, we can evaluate the quality of a given sensor configuration by computing the corresponding objective function of interest.

To begin, we first compare the OED solution using the greedy algorithm to random sensor selection; here, we generate 5,000 unique random designs with a budget of 10 sensors and evaluate the corresponding robust objectives (see (15) and (16)). Figure 4 presents the histograms of the objective function values alongside the OED solutions using the risk-neutral (a) and minimax objectives (b). From Figure 4, one can see the optimal design determined via the greedy algorithm outperforms random sensor selection.

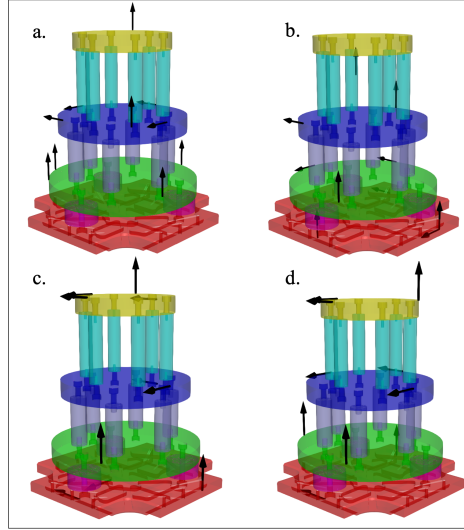


**Fig. 4** Histogram comparison of 5,000 random experimental designs versus the optimal design computed using the greedy algorithm for a) the risk-neutral objective given in (15) and b) the minimax objective given in (16).

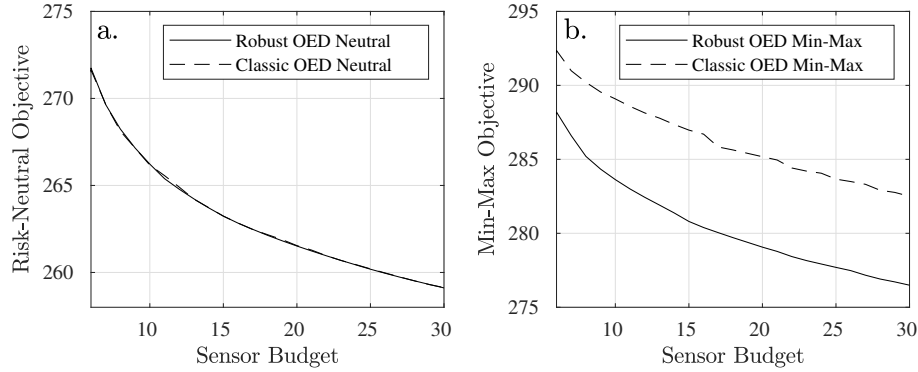
Next, we compare the performance of the proposed robust OED approach, which accounts for uncertainties in auxiliary parameters, to standard OED formulations. To do so, we consider that the standard OED formulation utilizes an average (with respect to material uncertainty) FRF matrix. To ensure a fair comparison, the FRF matrices are also evaluated at all the frequencies of interest (i.e. 100 Hz to 500 Hz at 8 Hz intervals).

First, consider the risk-neutral objectives. Figure 5 compares the standard (c) versus robust (d) designs for a budget of  $N = 10$  sensors using the risk-neutral objective. From Figure 5c and d, we see the optimal designs differ between robust and standard OED; to further explore this difference, Figure 6a shows the risk-neutral objective evaluated at the 1,000 FRF matrix validation samples as a function of sensor budget, comparing optimal robust versus standard designs. We see from Figure 6a a negligible difference in robust versus standard risk-neutral designs, indicating uncertainty in the auxiliary parameters does not have a significant impact on the OED problem in this case.

Next, Figure 5 compares the standard (a) versus robust (b) designs for a budget of  $N = 10$  sensors using the minimax objective. Unlike the risk-neutral case, we see significant differences in the designs. Furthermore, Figure 6b depicts the minimax objective over 1,000 FRF matrix samples as a function of sensor budget; here, we see that robust designs result in significantly lower object values, indicating that when tail statistics are of interest, auxiliary uncertainty impacts the OED problem. Note this result indicates that the impact of auxiliary uncertainty on the OED problem depends upon the objective formulation and goals of experimentation. More generally, the impact of such auxiliary uncertainties is dependent upon the model problem being considered, where in some cases (such our risk-neutral formulation) the impact is negligible-however, in other cases not properly accounting for auxiliary uncertainty may have significant impacts on the optimal design.



**Fig. 5** A comparison of robust versus standard experimental designs on the wedding cake geometry. Note the standard designs utilize the mean auxiliary parameters: a) standard minimax optimal design b) robust minimax optimal design, c) standard risk-neutral optimal design, d) robust risk-neutral optimal design.



**Fig. 6** Comparison of robust versus standard (e.g. classical) designs for the OED objective (averaged over 1,000 FRF matrix samples) versus the sensor budget: a) risk-neutral objective and b) minimax objective.

### *Average prediction variance greedy and convex optimization investigation*

We now turn our attention to the performance of the convex optimization OED formulation given in (6). Here, we focus on experimental designs that minimize model prediction uncertainty (as opposed to inference parameter uncertainty), where the variance of the prediction error at all model DoFs is used to quantify prediction uncertainty. Prediction error is defined by the difference between the predicted response  $f(\tilde{\theta}, \xi; d_j)$  and the model response  $f(\theta, \xi; d_j)$  conditioned on the auxiliary parameter uncertainty  $\xi$  for all candidate DoFs given by  $d_j \in \mathcal{D}$ . The predicted response is the model response using the estimated regression parameters  $\tilde{\theta}$  obtained from the least squares solution of the inverse problem. Note that  $\tilde{\theta}$  is a function of the chosen experiment design and the measurement uncertainty.

Since the estimator is unbiased, the variance of the prediction error at the  $j^{\text{th}}$  DoF is equivalent to the prediction variance defined as  $\sigma_{\tilde{y}_j}^2 \equiv \mathbb{E} \left[ f(\tilde{\theta}, \xi; d_j)^2 \right]$ , where the expectation is taken with respect to the measurement noise. We then refer to the average prediction variance as the sample approximated average over the candidate DoFs and over the auxiliary parameter realizations given by

$$\frac{1}{n_a n} \sum_{k=1}^{n_a} \sum_{j=1}^n \sigma_{\tilde{y}_j}^2(\xi_k, d), \quad (17)$$

where  $n$  is the total number of DoFs, and  $\sigma_{\tilde{y}_j}^2$  represents the prediction variance at the  $j^{\text{th}}$  DoF for the  $k^{\text{th}}$  sample of  $\xi$ .

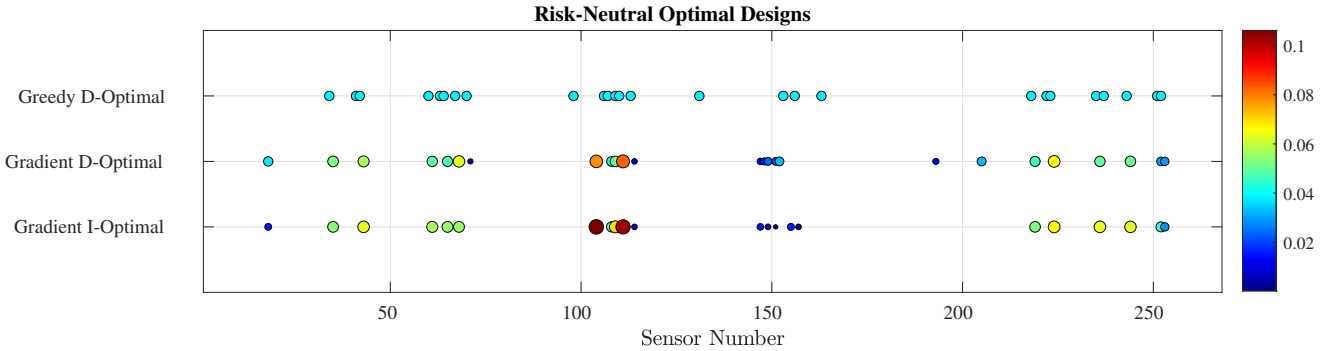


In this section, we consider the gradient-based optimization algorithm where  $\Psi(\mathbf{d}, \boldsymbol{\xi})$  is given by risk-neutral formulations of the  $D$ - and  $I$ -optimal criteria.  $I$ -optimality seeks to directly minimize the average prediction variance over all candidate experiments. The risk-neutral robust formulation of the  $I$ -optimal objective is approximated as

$$U(\mathbf{d}) \approx \frac{1}{n_a} \sum_{k=1}^{n_a} \text{Trace}(H(\boldsymbol{\xi}_k)C(\mathbf{d}, \boldsymbol{\xi}_k)H(\boldsymbol{\xi}_k)^*), \quad \boldsymbol{\xi}_k \sim \pi(\boldsymbol{\xi}), \quad (18)$$

where  $*$  represents the complex conjugate transpose. In contrast,  $G$ -optimality seeks to minimize the worst-case prediction variance. However, in the case of homoscedastic noise and a fixed set of auxiliary parameters,  $D$ - and  $G$ -optimality are equivalent [17], and hence we consider the objective given in (15).

In this study, we restrict the OED problem to 50 FRF matrix (i.e. auxiliary parameter) samples. To begin, we first compare the optimal designs produced by the convex optimization formulation and the greedy algorithm for the risk-neutral objective. The risk-neutral gradient-based  $I$ -optimal design and  $D$ -optimal design result in 26 and 22 non-zero sensor weights, respectively, and we select the first 26 sensors of the risk-neutral greedy  $D$ -optimal design. Figure 7 compares the optimal designs across criteria and optimization approach employed. In Figure 7, we see the weights corresponding to each candidate sensor location, where the size and color intensity of the marker reflect the magnitude of the sensor weight. Sensors selected by the greedy algorithm are assigned uniform weights. From Figure 7, we see that across varying criteria and optimization approaches, the optimal designs produced are similar.

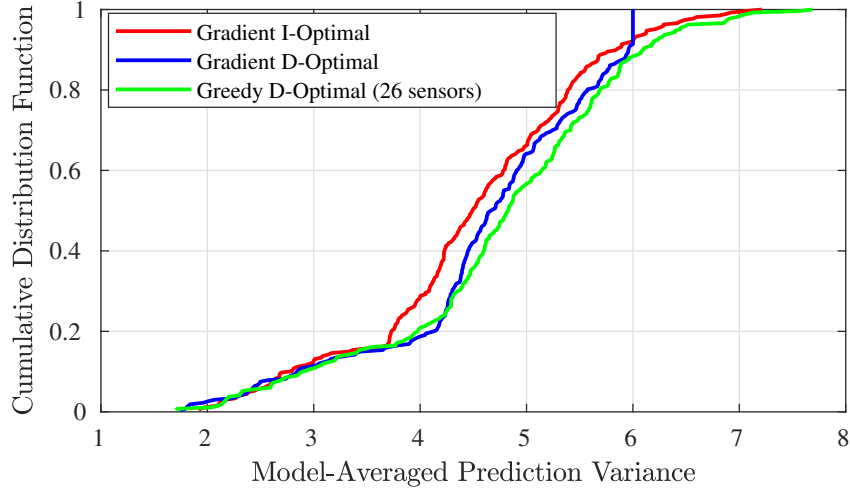


**Fig. 7** Optimal sensor designs using  $D$ - and  $I$ -optimality using convex optimization versus  $D$ -optimality computed using a greedy algorithm. Color intensity and marker sizes correspond to the design weight. Non-zero designs of the greedy solution are assigned uniform weights.

Now we proceed to verifying that the prediction variance aligns with the established design criteria. Figure 8 presents the cumulative distribution function (CDF) of the prediction variance across model DoFs averaged over samples of the auxiliary parameters (*Model-Averaged Prediction Variance*) and normalized by the variance of the measurement noise. In this example, the model-averaged prediction variance is computed over the 50 auxiliary parameter samples used to solve the robust OED, and the prediction variance per model sample is sample-approximated using 1,000 realizations of the estimated regression parameters  $\tilde{\boldsymbol{\theta}}$ . Each realization of  $\tilde{\boldsymbol{\theta}}$  involves generating a sample of the noise-perturbed response at the experiment design under consideration and solving an inverse problem.

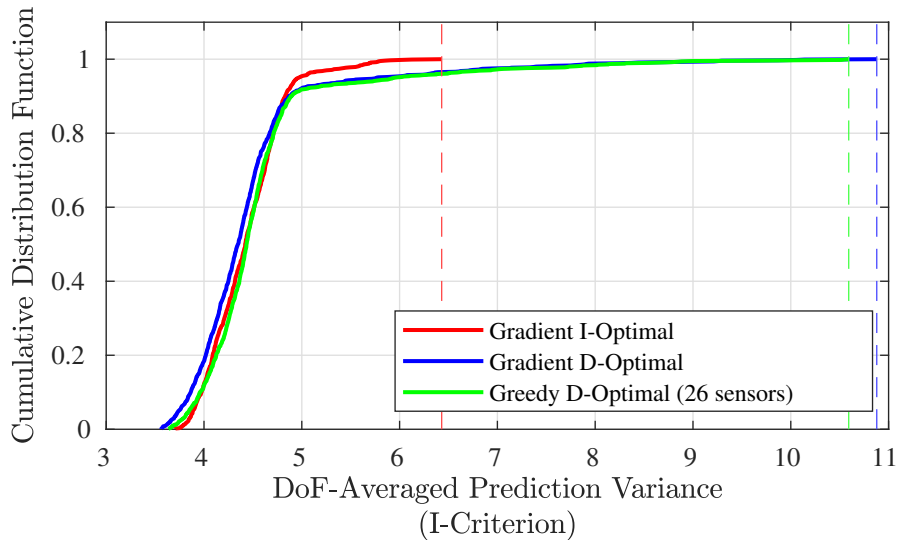
From the CDF curves in Figure 8, we observe that the prediction variance aligns with the established design criteria. Specifically, the gradient-based  $D$ -optimal design minimizes the maximum model-averaged prediction variance (CDF = 1), while the gradient-based  $I$ -optimal design minimizes the average model-averaged prediction variance. In general, there is a notable trade-off between these designs, where for example, minimal average prediction variance comes at the expense of a larger maximum prediction variance and vice-versa. Finally, we observe that the model-averaged prediction variances associated with the greedy  $D$ -optimal design tend to exceed those obtained from the gradient-based approaches.

Now, let's assume that the goal of the experiment design is to minimize the average prediction variance for any realization of the auxiliary parameters. To this end, we use the risk-neutral  $I$ -optimal OED formulation and compare its performance in terms of average prediction variance to the greedy and gradient-based  $D$ -optimal designs. For this study, we consider the average prediction variance given by (17) normalized with respect to the additive measurement noise variance  $\sigma^2$ . The sample-approximated prediction variances are generated using the same sampling method described previously, with the exception that we draw 1,000 samples of the auxiliary parameters. The corresponding average prediction variances are (i)  $4.48\sigma^2$  for gradient-based  $I$ -optimality, (ii)  $4.62\sigma^2$  for gradient-based  $D$ -optimality, and (iii)  $4.75\sigma^2$  for greedy-based  $D$ -optimality. Only a small improvement in average prediction variance using the  $I$ -optimal solution is observed.



**Fig. 8** Cumulative distribution function of the model-averaged prediction variance using the gradient-based  $D$ - and  $I$ -optimal designs and the greedy-based  $D$ -design.

Another trend can be observed if we investigate prediction variance averaged over the DoFs for each auxiliary parameter sample (*DoF-Averaged Prediction Variance*). Note that the DoF-averaged prediction variance is equivalent to the  $I$ -criterion. Figure 9 presents the CDF of the DoF-averaged prediction variance for the three design cases, using 1,000 auxiliary parameter samples. Despite the similarity in average prediction variance between  $D$ -optimal designs and the  $I$ -optimal design, the  $D$ -optimal designs result in long tails in the DoF-averaged prediction variance distribution. In other words, for some realizations of the auxiliary parameters, the DoF-averaged prediction variance using the  $D$ -optimal design is significantly worse than the  $I$ -optimal design.

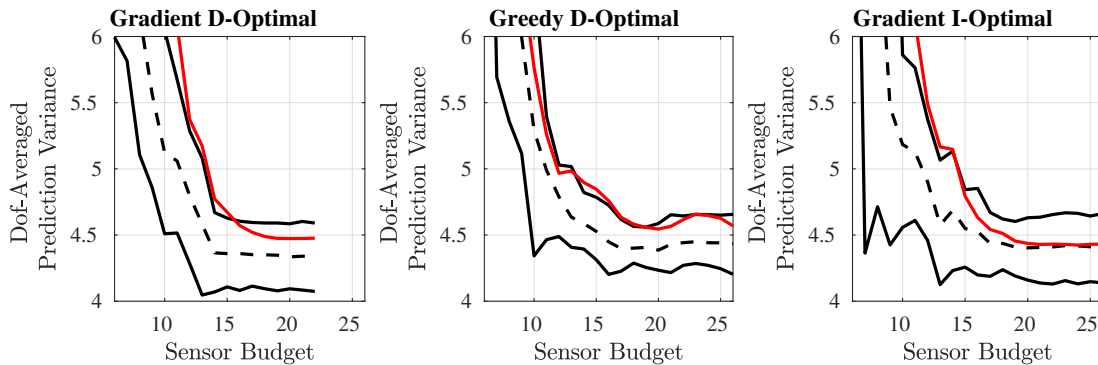


**Fig. 9** Cumulative distribution function of the DoF-averaged prediction variance ( $I$ -criterion) using the gradient-based  $D$ - and  $I$ -optimal designs and the greedy-based  $D$ -optimal design. Dashed lines indicate maxima for additional clarity.

Finally, we investigate a heuristic for down-selecting optimally selected experiments in order to enforce a sensor budget. The presented gradient-based optimization approach does not naturally incorporate a sensor budget  $N$ , so instead we consider one heuristic, which sorts the selected sensors by design weight, selecting the first  $N$  largest weights, and then rounding all other weights to zero. In this method, we then scale the non-zero weights so that they sum to one, adhering to their interpretation as probability masses.



Let's now compare the designs (greedy  $D$ -optimal versus gradient-based  $D$ - and  $I$ -optimal) by observing the sample-approximated, DoF-averaged prediction variance ( $I$ -criterion) as a function of sensor budget from 6 sensors to 26 sensors. Figure 10 depicts the 25<sup>th</sup> and 75<sup>th</sup> quantiles (solid black lines), the median (dashed black line), and the average (solid red line) of the DoF-averaged prediction variance across 1,000 realizations of  $\xi$  using the three design formulations. Note for the gradient-based  $D$ -optimal design, we terminated the study with 22 sensors because the optimization algorithm yielded only 22 non-zero weights. Upon inspection, it appears that the performance of the gradient-based  $I$ - and  $D$ -optimal designs is preserved down to about 19 and 17 sensors, respectively. The greedy-based  $D$ -optimal solution increases in prediction variance between 20 and 26 sensors, thus indicating that the prediction variance does not monotonically decrease with additional sensors. Generally speaking, the proposed heuristic appears to be a viable strategy for enforcing a sensor budget, though further investigation is needed.



**Fig. 10** The 25<sup>th</sup> and 75<sup>th</sup> quantiles (solid black line), the median (dashed black line), and the average (solid red line) of the DoF-averaged prediction variance ( $I$ -criterion) computed over all auxiliary parameter samples corresponding to the a) gradient  $D$ -optimal design, b) greedy  $D$ -optimal design, and c) gradient  $I$ -optimal design as a function of sensor budget.

## Conclusions

This work considered robust OED strategies that account for uncertainty in auxiliary model parameters when estimating an optimal experimental design and applied the approach to a structural dynamics force inversion problem. Uncertainty in auxiliary parameters was characterized probabilistically. The OED objectives were then formulated using minimax and expectation-based (i.e. risk-neutral) approaches, which compute respectively, the worst-case and expected determinant ( $D$ -optimal) of the asymptotic covariance or prediction variance ( $I$ -optimal). We investigated the performance of the proposed methods on a force inversion problem using measured vibration response in the presence of material uncertainty. A reduced basis method was used to construct a reduced order model of the expensive high-fidelity simulation, enabling computational tractability of the robust OED scheme. We further compared two approaches for optimizing the OED objective: 1) a greedy approach to sequentially determine optimal sensors and 2) a gradient-based approach that assigns probability masses to candidate sensor locations. We found that accounting for uncertainty when designing experiments resulted in a significant improvement in designs when considering the minimax objective function; however, a negligible improvement was noted for risk-neutral objective formulations. We note that such a result is model dependent, and cannot be generalized beyond the context of the presented computational example. Furthermore, in comparing gradient-based and greedy optimization approaches, similar optimal designs resulted. Finally, we found that the risk-neutral  $I$ -optimal gradient-based approach was most effective for minimizing the average model prediction uncertainty. In future work, we plan to study the minimax gradient-based optimization formulation and further investigate methods that minimize prediction uncertainty.

**Acknowledgments** This work was supported by the Laboratory Directed Research and Development program at Sandia National Laboratories, a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia LLC, a wholly owned subsidiary of Honeywell International Inc. for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

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## Chapter 14

# Sigma-point Bootstrapping for Uncertainty Quantification in Subspace Identification for Non-Gaussian Estimates

Szymon Greś, Charikleia Stoura, Vasilis Dertimanis, and Eleni Chatzi

**Abstract** The estimation of modal properties of dynamic systems is a fundamental task in engineering. Since the estimates are inherently afflicted with uncertainties due to the unknown ambient excitation, measurement noise, and limited data length, quantification of the related estimation quality is paramount for applications to, e.g., Operational Modal Analysis and Structural Health Monitoring. In recent years, uncertainty quantification methods for subspace-based modal parameter estimates have been proposed based on the first-order sensitivity-based covariance propagation. Although proven successful in multiple applications, the first-order framework relies on the linearization of the modal estimates and is inaccurate when the estimates are non-Gaussian. In this paper, we propose a new method for covariance propagation that does not rely on Taylor series expansion and provides accurate probability distribution approximation for non-Gaussian modal estimates, e.g., when obtained from short data sets. The proposed approach is based on an unscented transformation of data sample covariance, which is efficiently used in a bootstrap-like approach for subspace identification. The performance of the method is illustrated on a simple mechanical system.

**Keywords** Unscented transform · uncertainty propagation · subspace identification · non-Gaussian estimates · short data lengths

## Introduction

Covariance propagation and uncertainty quantification are fundamental tasks in modern engineering sciences. In recent years, uncertainty quantification methods for subspace identification methods have been proposed [1–4] based on the statistical delta method [5]. In this framework, the covariance of the estimated parameters is directly linked to the sample covariance of the vectorized data correlations through sensitivities obtained with the Taylor expansion of the considered identification algorithm. The first-order Taylor expansion provides an approximate method to propagate the covariance through a nonlinear function. Although the linearization is easy to implement, its precision is of the first-order, which by design does not encapsulate the high-order statistical information, whose significance is important for non-Gaussian estimates resulting from subspace identification problems with short data lengths or when estimated variables are close to the border of their distribution functions. While the second-order delta method can be used to approximate the asymptotic properties of the estimates at the borders [6, 7], it requires the knowledge of their true values, which limits its applications.

The unscented transformation is often used to approximate the mean and covariance of stochastic variables in nonlinear Bayesian filters. The key idea behind it is that with a fixed number of samples, the so-called sigma points, the estimate of a posterior mean and covariance of a variable is more accurate than using an approximation based on linearization or an arbitrary nonlinear transformation. Initially developed for Gaussian variables [8], which are fully characterized by their first two moments, it was found that the accuracy of the unscented transformation is reduced if the Gaussian assumption is not satisfied and the transformed function is highly nonlinear [9]. This sparked the development

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Szymon Greś

Department of Electronic Systems, Aalborg University, 9220 Aalborg, Denmark

e-mail: [sgres@es.aau.dk](mailto:sgres@es.aau.dk)

Charikleia Stoura · Vasilis Dertimanis · Eleni Chatzi

Department of Civil, Environmental and Geomatic Engineering, ETH Zürich, 8093 Zürich, Switzerland

e-mail: [charikleia.stoura@ibk.baug.ethz.ch](mailto:charikleia.stoura@ibk.baug.ethz.ch); [v.derti@ibk.baug.ethz.ch](mailto:v.derti@ibk.baug.ethz.ch); [chatzi@ibk.baug.ethz.ch](mailto:chatzi@ibk.baug.ethz.ch)

of unscented transforms that account for higher-order moments, like the skewness and kurtosis [10, 11]. While the unscented transformation is mostly applied in the Bayesian filtering context [12–14], its application relates to uncertainty quantification in general, where it can be applied to, e.g., uncertainty propagation in parameters of thermodynamic models [15].

In this work, we explore an application of the classic unscented transformation in the context of subspace identification to propagate the covariance of the sample covariance onto the estimates of modal parameters of a linear time-invariant (LTI) dynamic system. To this end, a set sigma points is constructed, the focus of which is to evaluate the considered random variable at the tail of its distribution. The chosen subspace identification algorithm is then applied to the set, and the unscented estimate is obtained by computing the mean of the transformed sigma points. Subsequently, its statistics are obtained using a kernel density estimate.

The paper is organized as follows: firstly, some background on the unscented transformation and subspace identification is given. Secondly, the sigma point uncertainty propagation framework is developed. Subsequently, the proposed method is validated on a simple numerical system.

## Background

The purpose of this section is to first recap the main steps of a classic unscented transformation of a random variable and then recall the first steps of a generic subspace identification algorithm; for a thorough summary, the interested reader is referred to [9] and [16] for the unscented transform and subspace identification respectively.

## Unscented Transformation

Let  $\hat{x} = g(\hat{z}) \in \mathbb{R}^s$  denote an outcome of a nonlinear transformation  $g(\cdot)$  of a random vector  $\hat{z} \in \mathbb{R}^s$  with  $z = \mathbb{E}[\hat{z}]$  and  $\Sigma_z = \mathbb{E}[(\hat{z} - z)(\hat{z} - z)^T]$  respectively denoting the mean and the covariance of  $\hat{z}$ . The sample mean and sample covariance of  $\hat{x}$  can be obtained with an unscented transform; to this end,  $2s$  sigma points are defined

$$\begin{aligned}\chi_i &= z + \bar{z}_i, \quad i = 1 \dots 2s \\ \bar{z}_i &= \sqrt{s} \left( \Sigma_z^{1/2} \right)_i, \quad i = 1 \dots s \\ \bar{z}_{i+s} &= -\sqrt{s} \left( \Sigma_z^{1/2} \right)_i, \quad i = 1 \dots s,\end{aligned}$$

where  $\chi_i$  denotes the  $i$ -th sigma point, the operator  $(\cdot)_i$  denotes an  $i$ -th column of a matrix and  $(\cdot)^{1/2}$  denotes a matrix square root. The sigma points are subsequently transformed

$$\mathcal{X}_i = g(\chi_i) \tag{1}$$

and the sample mean  $\bar{x}$  and sample covariance  $\Sigma_x$  of the transformed sigma points writes

$$\bar{x} = \frac{1}{2s} \sum_{i=1}^{2s} \mathcal{X}_i, \quad \Sigma_x = \frac{1}{2s} \sum_{i=1}^{2s} (\mathcal{X}_i - \bar{x})(\mathcal{X}_i - \bar{x})^T.$$

Both  $\bar{x}$  and  $\Sigma_x$  accurately approximate the first two moments of  $x$  when  $z$  is asymptotically Gaussian [9]. For non-Gaussian  $z$ , approximations are accurate to at least the second-order [8].

## Subspace Identification

Let  $\{y_k\}_{k=1, \dots, N+p+q}$ ,  $y_k \in \mathbb{R}^r$ , denote samples of a response of an LTI mechanical system collected by  $r$  sensors. We assume that this response is generated by an independent and identically distributed zero-mean input process of finite fourth moments, which is persistently exciting the system modes. The dynamic system is also assumed to be stable, observable, and with distinct eigenvalues. The input measurements are assumed not available; however, the main contribution of this work can easily be generalized to input-output subspace methods.

**Definition 1 (Data matrix)** Let  $a_k \in \mathbb{R}^b$  be a discrete signal at time step  $k$ . The parameter  $p$  defines the ‘past’ and ‘future’ data horizons. For  $0 \leq i \leq j \leq 2p - 1$  the data matrix  $\mathcal{A}_{i|j}$  writes

$$\mathcal{A}_{i|j} = \frac{1}{\sqrt{N}} \begin{bmatrix} a_i & a_{i+1} & \dots & a_{i+N-1} \\ a_{i+1} & a_{i+2} & \dots & a_{i+N} \\ \vdots & \vdots & \ddots & \vdots \\ a_j & a_{j+1} & \dots & a_{j+N-1} \end{bmatrix} \in \mathbb{R}^{(j-i+1)b \times N}. \quad (2)$$

The first step in subspace identification is the estimation of the data projection matrix  $\mathcal{H}$ . To this end, we define data matrices  $\mathcal{Y}^- = \mathcal{Y}_{0|p-1} \in \mathbb{R}^{ps \times N}$ ,  $\mathcal{Y}^+ = \mathcal{Y}_{p|2p-1} \in \mathbb{R}^{ps \times N}$ , where  $p$  denotes the time lag. The shape of  $\hat{\mathcal{H}}$  depends on the chosen projection to nullify the influence of inputs and the instrumental variable to eliminate the noise. Hereafter, as an example, we use the output-only covariance-driven subspace identification algorithm with the projection matrix

$$\hat{\mathcal{H}} = \mathcal{Y}^+ (\mathcal{Y}^-)^T. \quad (3)$$

For a chosen model order, the estimates of modal parameters are obtained as a function of  $\hat{\mathcal{H}}$

$$\hat{\eta} = g(\hat{\mathcal{H}}), \text{ with } \hat{\eta} = \begin{bmatrix} \hat{f} \\ \hat{\zeta} \\ \text{vec}(\hat{\varphi}) \end{bmatrix}$$

where  $\hat{f}$ ,  $\hat{\zeta}$ , and  $\hat{\varphi}$  are the estimated natural frequencies, damping ratios, and mode shapes, respectively.  $g(\cdot)$  denotes the chosen subspace identification algorithm.

## Sigma point bootstrap for uncertainty propagation

In this section, an application of the unscented transformation for uncertainty propagation in subspace identification is outlined. Let  $\hat{\Sigma}_{\mathcal{H}}$  be a consistent estimate of the sample covariance of the vectorized  $\hat{\mathcal{H}}$  obtained from  $n_b$  blocks of available data

$$\hat{\Sigma}_{\mathcal{H}} = \frac{1}{n_b(n_b - 1)} \sum_{j=1}^{n_b} \text{vec}(\hat{\mathcal{H}}_j - \hat{\mathcal{H}}) \text{vec}(\hat{\mathcal{H}}_j - \hat{\mathcal{H}})^T, \quad (4)$$

where  $\hat{\mathcal{H}}_j$  denotes a Hankel matrix computed on  $j = 1 \dots n_b$  data blocks. The sigma points are defined as

$$\begin{aligned} \hat{\mathcal{H}}_i &= \hat{\mathcal{H}} + \tilde{\mathcal{H}}_i, \quad i = 1 \dots 2n_b \\ \tilde{\mathcal{H}}_i &= \sqrt{k_i} \left( \Sigma_z^{1/2} \right)_i, \quad i = 1 \dots n_b \\ \tilde{\mathcal{H}}_{i+n_b} &= -\sqrt{k_i} \left( \Sigma_z^{1/2} \right)_i, \quad i = 1 \dots n_b, \end{aligned}$$

where  $k_i$  denotes an  $i$ -th element of  $k = [1 \ 3 \ \dots \ 2n_b]$  and  $\tilde{\mathcal{H}}_i = \text{vec}^{-1}(\tilde{\mathcal{H}}_i)$  where  $\text{vec}^{-1}(\cdot)$  denotes an inverse vectorization operator. Note that the weighting strategy employed herein aims at evaluating the sigma points at the tail of the sample distribution of  $\hat{\mathcal{H}}$ , which in principle is different than in the classic unscented transformation. Once the samples of the modal parameters are obtained from the transformed sigma points

$$\hat{\eta}_i = g(\hat{\mathcal{H}}_i),$$

the unscented estimate can be computed

$$\hat{\eta}^{\text{un}} = \frac{1}{2n_b} \sum_{i=1}^{2n_b} \hat{\eta}_i$$

and the joint probability distribution of the unscented estimates can be established through the kernel density estimation

$$\hat{f}(\eta) = \frac{1}{n_p h} \sum_i^{n_p} K\left(\frac{\eta - \bar{\eta}}{h}\right),$$

where  $\bar{\eta}$  are equally-spaced points that cover the range of  $\hat{\eta}_i$  for  $i = 1 \dots 2n_b$ ,  $n_p$  is the number of the points,  $K$  is the kernel smoothing function, and  $h$  is the bandwidth. In this work, a Gaussian kernel with 100 data points is used.

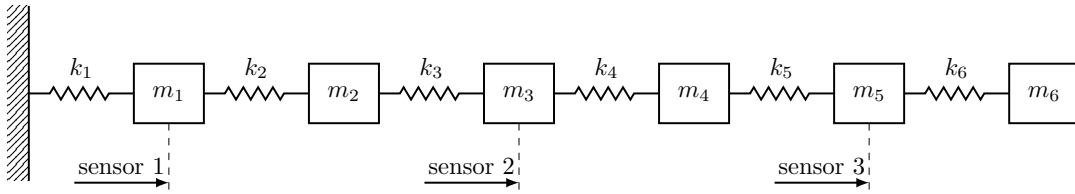
## Numerical Validation

The uncertainty propagation strategy outlined in the previous section is validated on a 6 DOF mechanical chain-like system that for any consistent set of units is modeled with spring stiffness  $k_1 = k_3 = k_5 = 100$  and  $k_2 = k_4 = k_6 = 200$ , mass  $m_i = 1/20$ , and a non-proportional damping matrix

$$C = \begin{bmatrix} 0.2007 & -0.0236 & 0.0595 & 0.1170 & 0.0736 & 0.1341 \\ -0.0236 & 0.0192 & -0.1054 & -0.0748 & -0.1617 & -0.0773 \\ 0.0595 & -0.1054 & 0.1263 & -0.0297 & -0.0843 & 0.0286 \\ 0.1170 & -0.0747 & -0.0297 & 0.2570 & -0.0748 & 0.1227 \\ 0.0735 & -0.1617 & -0.0842 & -0.0748 & -0.0601 & -0.1141 \\ 0.1341 & -0.0773 & 0.0286 & 0.1226 & -0.1142 & 0.2558 \end{bmatrix} \cdot 10^{-4}.$$

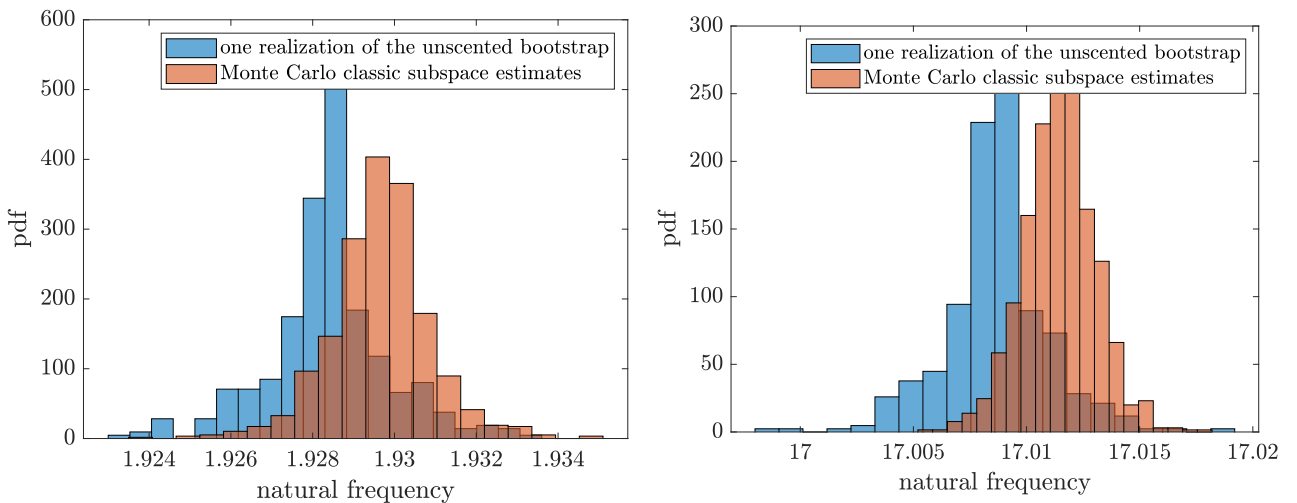
The chain is illustrated in Figure 1.

The system is excited by a white noise signal persistently acting at all DOFs. The acceleration outputs are simulated at DOFs 1, 3, and 5 at a sampling frequency of 50 Hz for 200 seconds, generating 10.000 data points. An additional white measurement noise with 5% of the standard deviation of the output is added to each response measurement. The computations are performed in a Monte Carlo setup with 1000 realizations of the input-output data. The output-only covariance-driven subspace system identification is used, and the estimates are obtained at system order 12, using 15 time lags for the computation of the Hankel matrix.

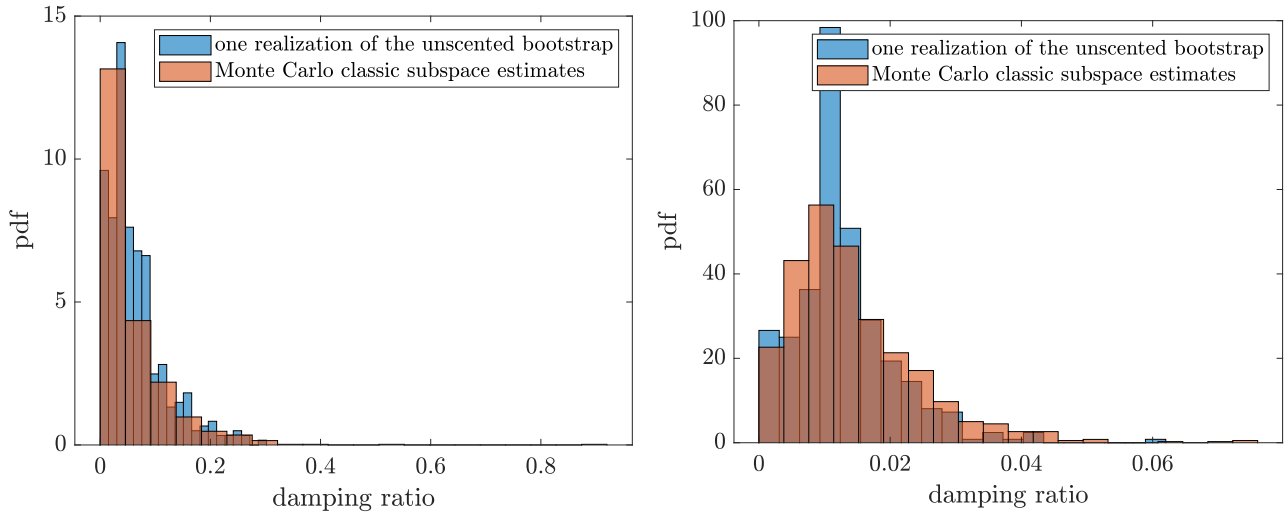


**Fig. 1** 6 DOF chain system sketch.

The sample covariance of the Hankel matrix is obtained by dividing the output data to 200 blocks, resulting in 400 sigma points for uncertainty propagation. The histograms of the classic subspace-based natural frequency and damping ratio estimates, and the corresponding estimates obtained from one realization of the sigma point sampling strategy are illustrated in Figures 2 and 3.



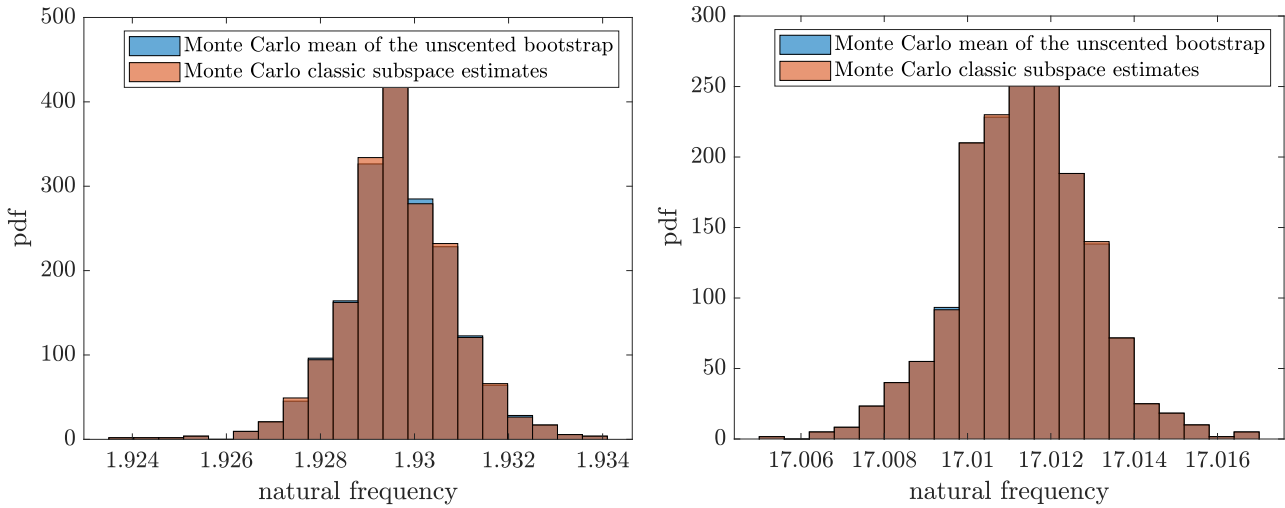
**Fig. 2** Natural frequency estimates-first mode (left) and sixth mode (right).



**Fig. 3** Damping ratio estimates-first mode (left) and sixth mode (right).

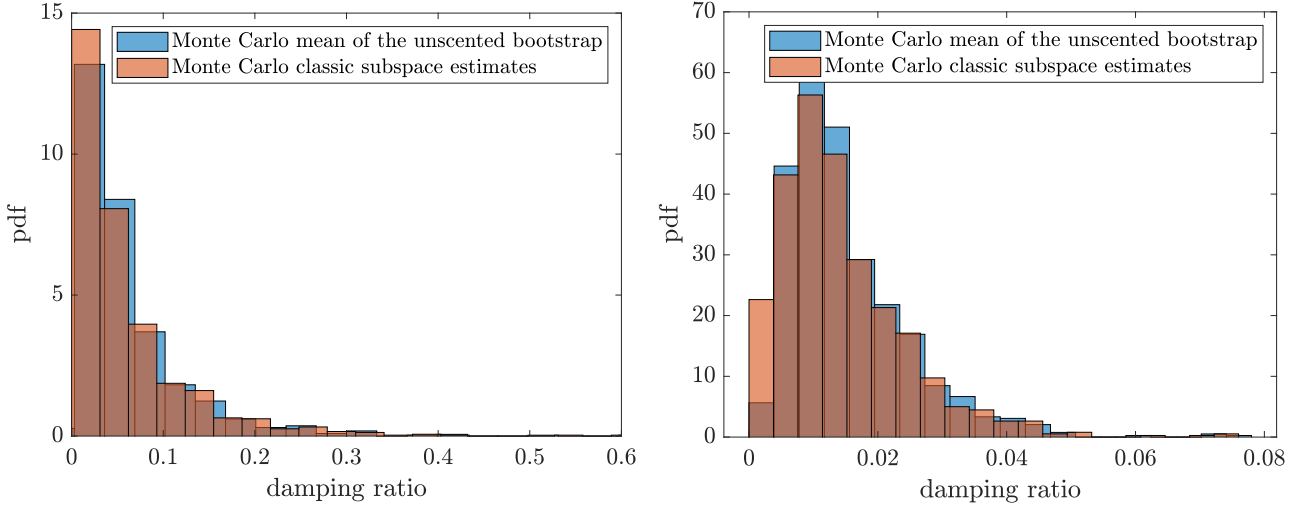
It can be viewed that the independent realizations of the natural frequency and damping ratio estimates yield histograms illustrating their distributions; while the shape of the natural frequency histograms for both approaches resembles a Gaussian probability density function, the damping ratio estimates accumulate at the border of the distribution at 0, where the Gaussian assumption is clearly violated. While it is expected that the classic first-order perturbation approach is suitable for uncertainty quantification of a Gaussian distributed variable, its application for uncertainty assessment of estimates that are non-Gaussian is not adequate.

Figure 2 illustrates that although the mean of the histogram corresponding to the classical subspace-based natural frequency estimates does not coincide with the mean of the histogram corresponding to one realization of the unscented bootstrap, the spread of the histograms is similar. It is not expected that the means will match in this case; a match between the two should be expected when the mean of the unscented estimates is considered, which is illustrated in Figures 4 and 5.

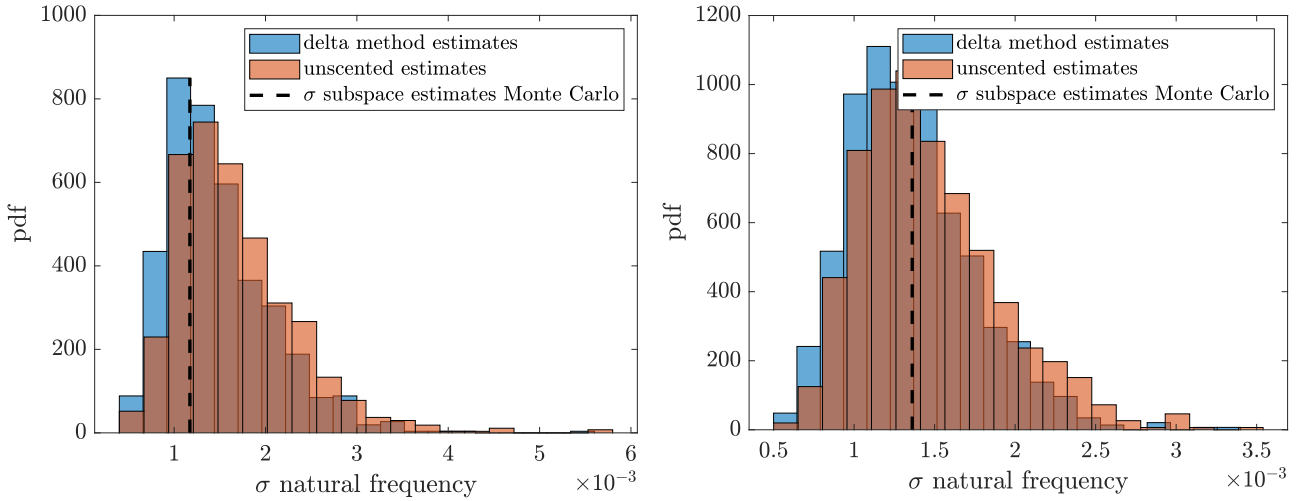


**Fig. 4** Natural frequency estimates-first mode (left) and sixth mode (right).

Figures 2 and 4 suggest that the natural frequency estimates are possibly asymptotically Gaussian distributed; therefore, hereafter, we will focus on comparing the standard deviations obtained using the classic first-order perturbation approach and the sample standard deviations obtained from the unscented bootstrap, omitting the kernel density estimation step. Histograms of both quantities are illustrated in Figure 6. It can be viewed that the mean of the standard deviation estimates obtained using the first-order perturbation approach and the sample standard deviations of the



**Fig. 5** Damping ratio estimates- the first mode (left), the sixth mode (right) .



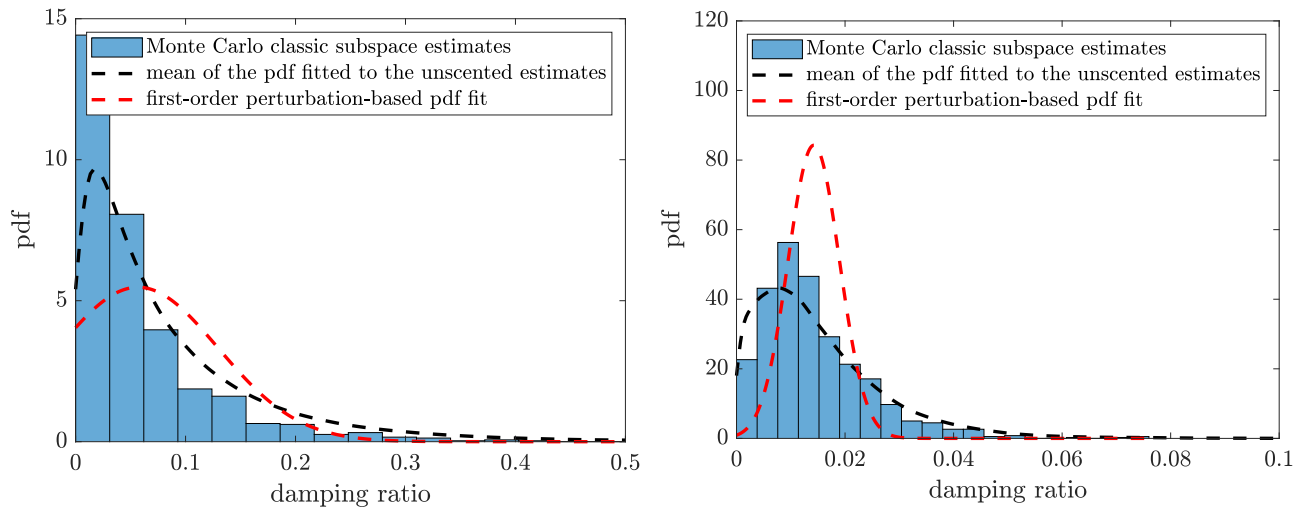
**Fig. 6** Standard deviation ( $\sigma$ ) of natural frequency estimates-first mode (left) and sixth mode (right).

unscented estimates match well with each other and coincide with the standard deviation of the classic subspace Monte Carlo histograms.

The motivation behind the proposed method is the estimation of the statistical properties of non-Gaussian subspace-based estimates. This is showcased in the estimates of damping ratios, which are clearly non-Gaussian due to the significant identification errors stemming from short data lengths. Therefore, to validate the proposed approach, the average of the probability density functions obtained with the first-order approach and the unscented bootstrap are compared to the histogram of the estimates in Figure 7. It can be viewed that the average probability density function obtained via kernel density estimate matches well with the histograms of the damping ratio estimates despite the estimates being non-Gaussian.

The proposed method clearly outperforms the first-order approach in uncertainty quantification of non-Gaussian variables; however, the price for the accuracy is the numerical efficiency of the bootstrap: using Matlab 2020b on AMD Ryzen 7 5800H CPU with 64GB of RAM, one evaluation of an efficient implementation of the first-order approach takes 0.01s and a parallelized unscented bootstrap takes 0.49s.





**Fig. 7** Uncertainty of damping ratio estimates-first mode (left) and sixth mode (right).

## Conclusions

In this work, we present an application of the unscented transformation for the estimation of the probability density functions of the modal parameter estimates obtained in subspace identification. The performance of the proposed method is equivalent to the classic first-order sensitivity-based approach for estimates following a Gaussian distribution. However, it is shown that classic methods fail to approximate the statistical properties of non-Gaussian parameters, whereas the proposed unscented transformation provides a more accurate approximation. The cost of this increased accuracy is a reduction in the computational efficiency of the bootstrap.

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# Chapter 15

## Vehicle Load Identification of Bridges Using Acceleration Response Data

Iman Dabbaghchian and Shamim Pakzad

**Abstract** Identifying the characteristics of passing vehicles on the bridge, often referred to as weigh-in-motion (WIM) systems, is of interest in various applications of bridge maintenance and monitoring. WIM systems enable the measurement of vehicle weights, including the weight of individual axles, depending on the accuracy of the system, as they travel over the pavement. Following years of development, WIM systems are now capable of delivering detailed traffic information, including vehicle classification, axle and wheel weights, total gross weight, speed, and transit date and time. WIM systems also contribute to enhancing the safety evaluations of pavements and bridges. However, depending on the type of WIM systems, several obstacles must be addressed for these systems to see widespread adoption. These challenges include their high cost, the requirement for specialized technical knowledge, limited durability, and the need for ongoing inspection and maintenance. This study introduces an innovative method that can replicate a WIM system utilizing acceleration response data. The acceleration data is relatively easy and inexpensive to collect. By employing signal processing techniques, specifically wavelet transforms, this approach aims to extract and decompose the response signal to accurately determine the number, weight, and speed of passing vehicles on the bridge. This study digs deep into the dynamics of the bridge and its interaction with vehicles through case studies of moving vehicle loads with different weight classes, speeds, and directions. The uncertainty in the prediction is also discussed in the results.

**Keywords** Bridge load identification · Acceleration response data · Wavelet transform · Bridge health monitoring · Signal processing

### Introduction

Structural health monitoring (SHM) of bridges is a critical task in ensuring the long-term safety, durability, and reliability of both the bridges themselves and the societies they serve. As vital components of transportation infrastructure, bridges not only facilitate the flow of goods and services but also play a significant role in economic and social stability. However, many of these structures are aging, particularly in the United States, where a large percentage of the nation's bridges were built more than 50 years ago. The American Society of Civil Engineers (ASCE) has reported that over 42% of U.S. bridges are at least 50 years old, with more than 46,000 deemed structurally deficient. These aging bridges are increasingly subject to material deterioration, environmental degradation, and heavier loads than originally designed for, which highlights the urgent need for continuous and precise monitoring systems.

The increasing volume of traffic, especially from overweight enforcement vehicles, exacerbates safety risks and presents additional maintenance challenges for transportation agencies, particularly on deteriorating bridge infrastructure or those under extreme cases. [1, 2] This can lead to significant damage to both the structural and non-structural components of the bridge, such as beams and pavements, causing issues like fatigue, cracking, and, in extreme cases, structural collapse. [3] Identifying and tracking the number of large vehicles and overall traffic volume can significantly assist bridge maintenance agencies in optimizing their maintenance strategies and ensuring the longevity and safety of the structure.

SHM systems rely on field measurement data, and dynamic acceleration response has been used for operational modal analysis (OMA) to extract modal characteristics such as natural frequencies, damping ratios, and mode shapes. These parameters provide insights into the global condition of the structure and are widely utilized in the assessment of bridge performance under both operational and extreme conditions [4, 5]. However, the use of accelerometers extends beyond

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Iman Dabbaghchian · Shamim Pakzad

Department of Civil and Environmental Engineering, Lehigh University, Bethlehem, PA 18015

e-mail: [imd220@lehigh.edu](mailto:imd220@lehigh.edu); [snp208@lehigh.edu](mailto:snp208@lehigh.edu)

modal analysis, with their ability to capture high-frequency content offering new opportunities to extract additional features related to dynamic loads. Recently machine learning methods have also retained a lot of attention, and multiple applications have been done with the combination of artificial neural network and acceleration data [6, 7, 8, 9, 10]

Recent advances in signal processing techniques, particularly the application of wavelet transforms, have provided sharper time-frequency resolution, enabling the identification of high-frequency content that is often obscured by noise [11, 12, 13]. Wavelet-based approaches have shown significant potential in capturing transient events, such as vehicle-induced loads, which traditional Fourier-based methods may fail to detect effectively, especially in noisy and non-stationary signals [14, 15]. Identifying the dynamic loads, such as the moving vehicles on bridges, provides valuable information about the structure's response to real-time loads. This information can be used to improve load modeling, enhance predictive maintenance strategies, and optimize load distribution for bridge management systems [16, 17]. Moreover, accurate identification of vehicle trajectories and weights can support the development of more robust load-rating systems, contributing to the structural integrity of bridges under varying traffic conditions.

In this paper, we present a novel framework for extracting vehicle trajectories on bridges by leveraging the high-frequency components of acceleration data derived from wavelet transform analysis. The proposed methodology effectively captures the dynamic interaction between vehicles and the bridge structure. The accuracy and robustness of the framework are validated through multiple numerical simulations, demonstrating its potential for real-time bridge monitoring and load identification.

## Methodology

The proposed methodology begins with the acquisition of acceleration data from multiple locations along the bridge structure. This data is then utilized to detect and characterize the dynamic input load, specifically aiming to identify or estimate critical parameters of the passing vehicles, such as their weight and speed. To demonstrate the effectiveness of this approach, a detailed case study is conducted using a numerical simulation of a two-dimensional steel truss bridge. This simulated model is based on a real-world bridge located in Pennsylvania, USA, providing a realistic and practical context for the analysis.

Let  $a_i(t)$  represent the acceleration response signal from location  $i$ , where  $t$  is time and  $i = 1, 2, \dots, M$  and the signals are obtained from  $M$  locations at a sampling frequency of  $f_s$  Hz. For each signal  $a_i(t)$ , we select a time window  $T_w$  of length  $\Delta t$  (e.g., 30 seconds) and compute its synchrosqueezed wavelet transform (SWT). The SWT produces wavelet coefficients  $W_i(t, f)$  as a function of time  $t$  and frequency  $f$ , which can be written as:

$$W_i(t, f) = \text{SWT}(a_i(t), f_s) \quad (1)$$

where  $W_i(t, f)$  represents the wavelet coefficients of the signal  $a_i(t)$  at time  $t$  and frequency  $f$ . Also, the frequency range for the wavelet coefficients will be from 0 to the Nyquist frequency which is  $f_s/2$ . However, the maximum frequency can be cut off to a lower value of interest, depending on the modal characteristics of the structure. The frequency range  $[0, f_{\max}]$  is then divided into  $N$  segments, each with a width of  $\Delta f = \frac{F_{\max}}{N}$ . For example, if  $f_{\max} = 20$  Hz and  $N = 8$ , the frequency bands are defined as:

$$F_n = [(n-1)\Delta f, n\Delta f], \quad n = 1, 2, \dots, N \quad (2)$$

Thus, for  $N = 8$ , the intervals are  $[0, 2.5], [2.5, 5], \dots, [17.5, 20]$  Hz. For each frequency band  $F_n$ , the wavelet coefficients  $W_i(t, f)$  are averaged across the frequency range to obtain a new time signal  $A_i^n(t)$ , defined as:

$$A_i^n(t) = \frac{1}{\Delta f} \int_{F_n} W_i(t, f) df \quad (3)$$

where  $A_i^n(t)$  represents the averaged wavelet coefficient for location  $i$  in the  $n$ -th frequency band. For each frequency band  $F_n$ , a moving average is applied to the wavelet coefficients  $A_i^n(t)$  using a window of length  $\Delta t_m$  (e.g., 1 second), with an overlapping shift  $\Delta s_m$  (e.g., 0.5 seconds) for smoothing purposes. The smoothed signal  $\bar{A}_i^n(t)$  is given by:

$$\bar{A}_i^n(t) = \frac{1}{\Delta t_m} \int_{t-\Delta t_m/2}^{t+\Delta t_m/2} A_i^n(\tau) d\tau \quad (4)$$

where  $\Delta t_m$  is the moving average window length, and  $\Delta s_m$  is the time shift applied between successive windows.

After processing the signal in the current time window, we shift the window by  $s$  seconds and repeat the analysis. If the initial window starts at  $t_0$ , the  $k$ -th shifted window will begin at  $t_0 + ks$ , where  $s$  is the shift step. The process is repeated for each time window, producing a series of smoothed wavelet coefficients for each frequency band.

$$\bar{A}_i^n(t_0 + ks) \quad \text{for } k = 0, 1, 2, \dots \quad (5)$$

By generating a series of smoothed wavelet coefficients for each frequency band and aligning the signals corresponding to each location with their respective positions on the structural joints (mapping the geometry), a heatmap can be produced that effectively visualizes the trajectory of vehicle movement across the bridge.

## Analysis

To illustrate the effectiveness of the proposed method, a numerical example is presented and the corresponding results are discussed in detail. The example involves a finite element analysis (FEA) of a two-dimensional bridge subjected to moving loads, which is representative of real-world conditions. The simulation uses standard AASHTO vehicle configurations to model the dynamic interaction between the bridge and the passing traffic. In the first scenario, three trucks are introduced: two traveling in the direction from sensor S1 to S16 at the speed of 18.3 m/s (60 ft/s), while the third truck moves in the opposite direction at the speed of 16.8 m/s (55 ft/s). This mixed traffic flow provides a robust test case to evaluate the response of the bridge under varying load conditions. Figure 1 illustrates the geometry of the 2D bridge model and presents the vertical acceleration response at each joint along the structure, as captured during the moving load analysis. By analyzing these acceleration responses, we can further assess the method's capability to detect and estimate the input parameters such as vehicle weight and speed, based on the bridge's dynamic response.

The collected responses from 16 locations have a sampling frequency of 100 Hz, which means the observability of the frequency content is 50 Hz. Figure 2 shows the magnitude of the wavelet coefficients. Even from these figures, it can be observed that the high-frequency content is moving across the spatial and temporal domain. The main contribution of this work is to introduce how these high-frequency contents can be traced and analyzed.

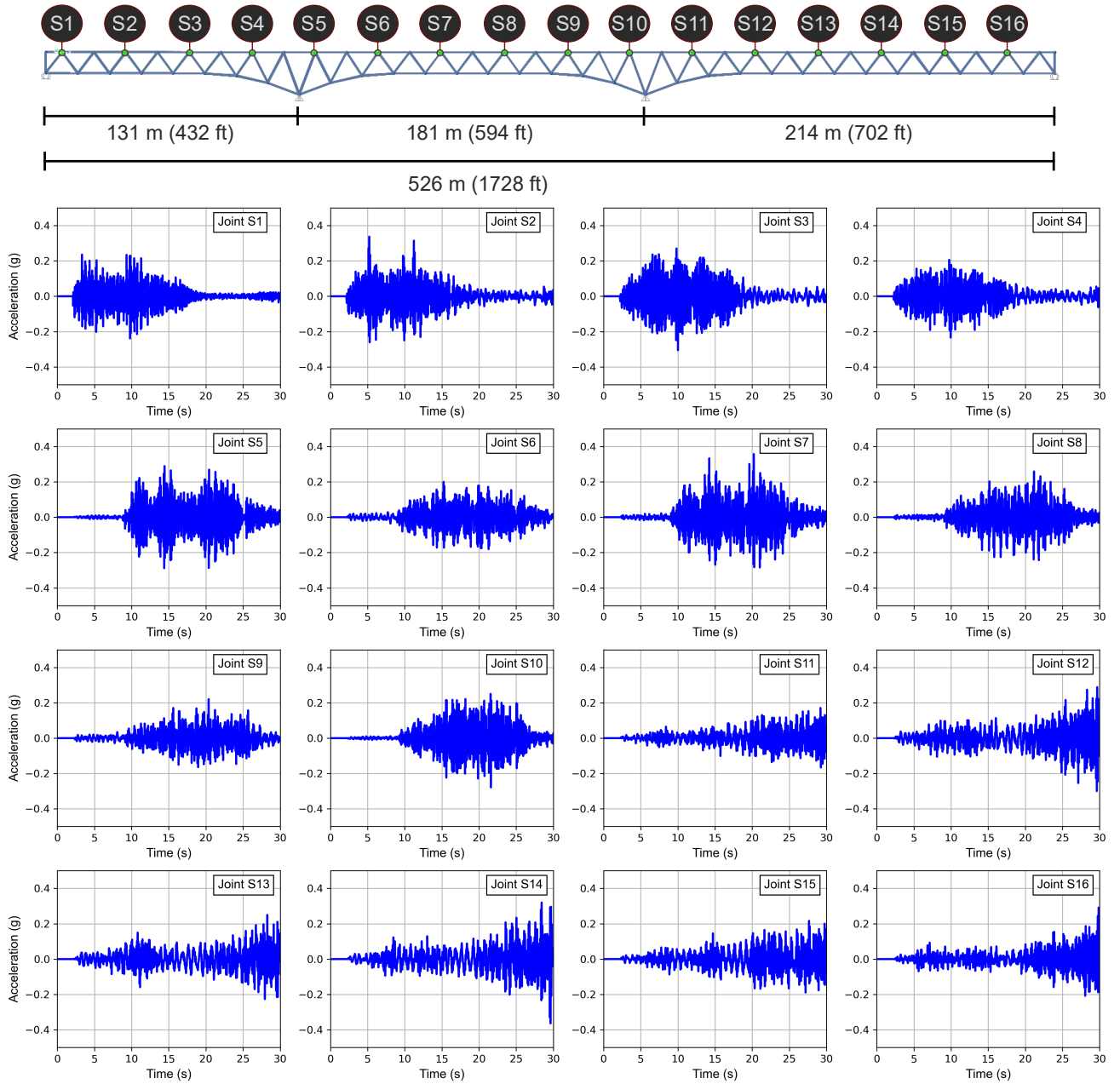
The synchrosqueezed wavelet transform [13] is a time-frequency analysis method that is useful for analyzing multicomponent signals with oscillating modes and was aimed to sharpen the time-frequency representation. The primary difference between the synchrosqueezed wavelet transform (SWT) and the regular wavelet transform lies in their time-frequency representation and the precision of frequency localization. The regular wavelet transform provides a time-frequency analysis where the frequency resolution decreases with higher frequencies, and the result is a multi-resolution analysis with non-uniform frequency bands. In contrast, the synchrosqueezed wavelet transform enhances the frequency resolution by "reassigning" the wavelet coefficients closer to their actual frequency. This process leads to a sharper and more concentrated time-frequency representation, which is particularly useful for separating closely spaced or overlapping frequency components in non-stationary signals. Essentially, the SWT aims to recover a finer frequency detail, which the regular wavelet transform lacks, making it a more precise tool for analyzing signals with varying frequencies over time.

Following the calculation of the wavelet magnitude coefficients, the time-frequency representation was divided into eight distinct frequency bands. For each band, the wavelet magnitudes were averaged, resulting in eight distinct averaged magnitude signals for each measurement location. To further refine these signals, a moving average smoothing technique, as detailed in the methodology section, was applied to reduce fluctuations in the wavelet magnitudes and produce smoother time-frequency representations. Subsequently, the averaged magnitude signals from the 16 sensor locations (S1 to S16) were spatially aligned to construct a series of spatial-temporal maps. Specifically, the signals from each location were placed adjacently, producing eight separate maps corresponding to the eight frequency bands (see Figure 3). In these maps, the x-axis represents time, while the y-axis corresponds to the spatial arrangement of the bridge, with locations S1 to S16 aligned according to the horizontal coordinates of the structural joints. The z-axis captures the average wavelet magnitudes.

This approach enables a clearer representation of both the spatial and temporal domains of the signal, allowing for a more comprehensive analysis of dynamic responses across the structure. The visualized heatmaps thus provide a detailed perspective on the variations in wavelet magnitudes across different frequency bands and locations, facilitating the identification of key patterns, such as vehicle movement trajectories on the bridge.

In Figure 3, the plots corresponding to the first five frequency bands (0 to 12.5 Hz), which encompass the frequency range associated with the first 20 structural modes, do not exhibit a discernible trajectory pattern for the passing vehicles. This indicates that within these lower frequency bands, the vehicle-induced dynamic responses are likely dominated by modal contributions that obscure any clear indication of the moving loads. In contrast, the plots corresponding to the higher frequency bands—specifically the last three bands—clearly reveal the trajectory of the vehicles as they traverse the bridge. These higher frequency bands capture the localized dynamic effects of the moving loads, which are less influenced by the global modal behavior, thereby allowing the vehicle trajectories to be distinctly visualized in the time-frequency domain.

The clear distinction between the results in the lower and higher frequency bands emphasizes the importance of selecting appropriate frequency ranges when conducting moving load analyses. The ability to capture the vehicle trajectory is crucial for accurately assessing the dynamic interaction between the vehicles and the structure. By focusing on the higher frequency

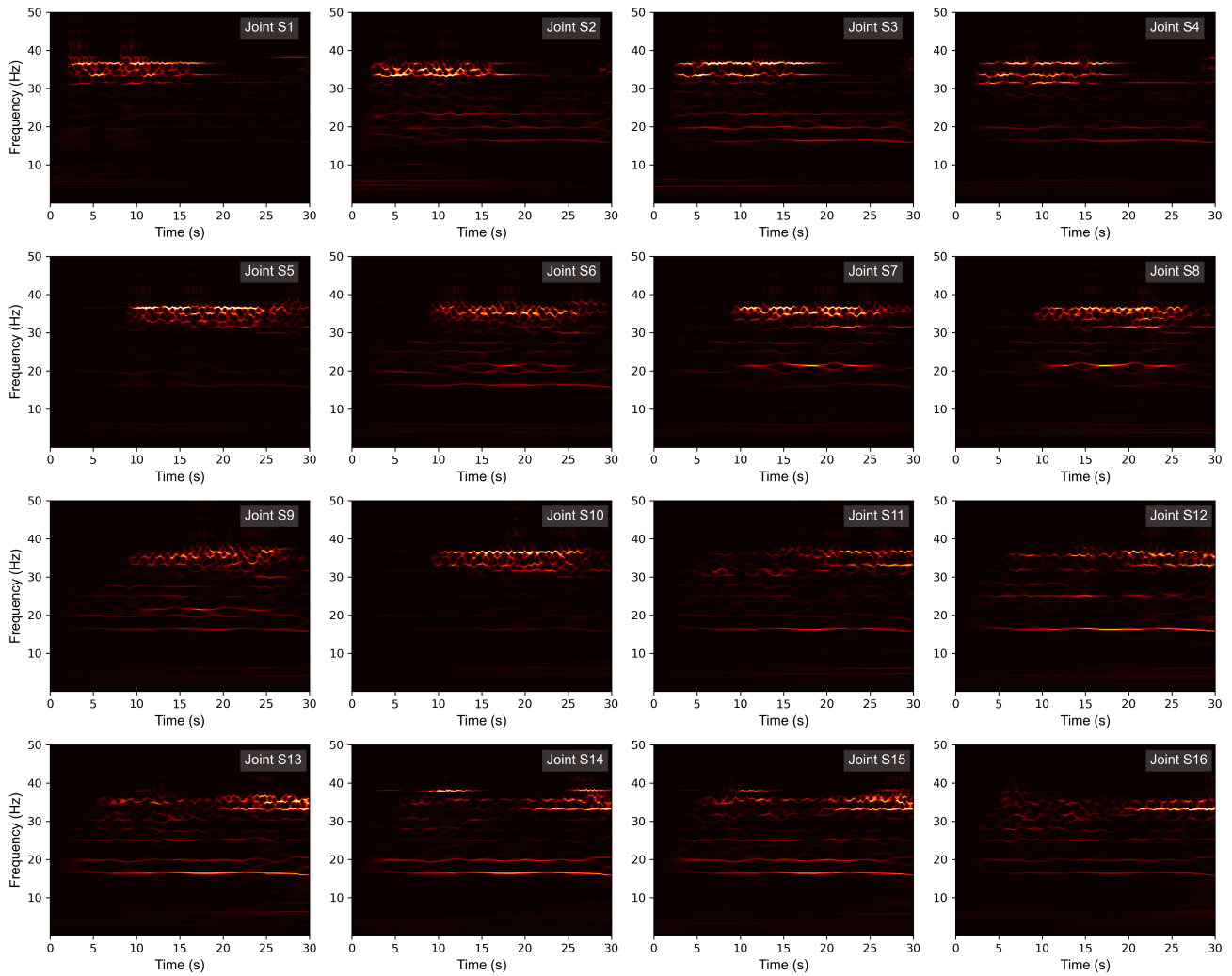


**Fig. 1** Vertical acceleration response signals from 16 joints along the bridge deck position under the moving load with three vehicles. The bridge geometry and joints coordinates are also shown at the top.

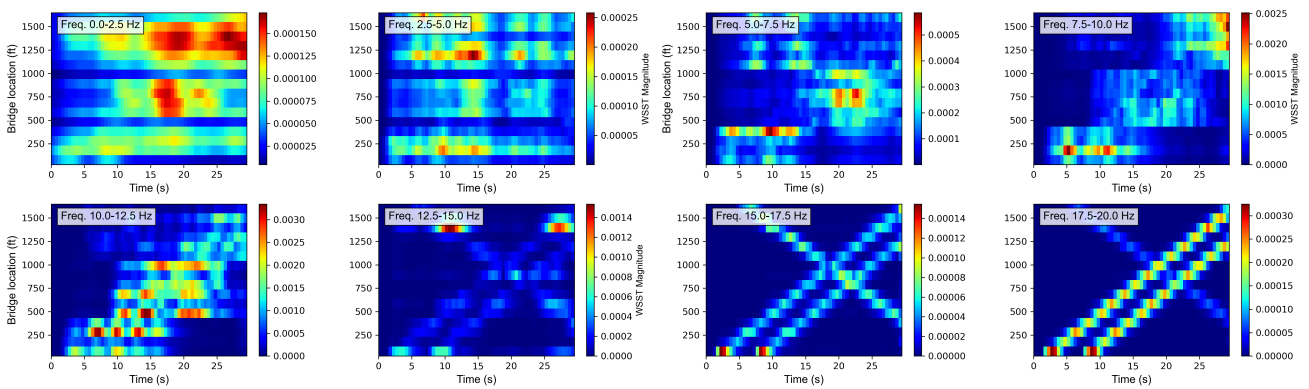
components, the trajectory maps provide valuable insights into the localized structural response to the moving loads, which is essential for understanding the dynamic behavior of the bridge under vehicular traffic.

The final frequency bin provides the most accurate estimation of vehicle trajectories, demonstrating that higher frequency bands yield better precision in identifying vehicle characteristics. In the spatio-temporal visualization of averaged wavelet magnitudes shown in Figure 3, the magnitude of the wavelet coefficients is indicative of the vehicle's weight, while the slope of the trajectory lines correlates with the vehicle's speed. Based on these observations, the final frequency bin was selected and further refined using interpolation techniques to enhance the resolution of the trajectory estimates. These refined estimates were then compared to the ground truth from the moving load analysis involving three vehicles, as presented in the top two plots of Figure 4. The results show a high degree of accuracy, with the vehicle trajectories being clearly and distinctly identified.



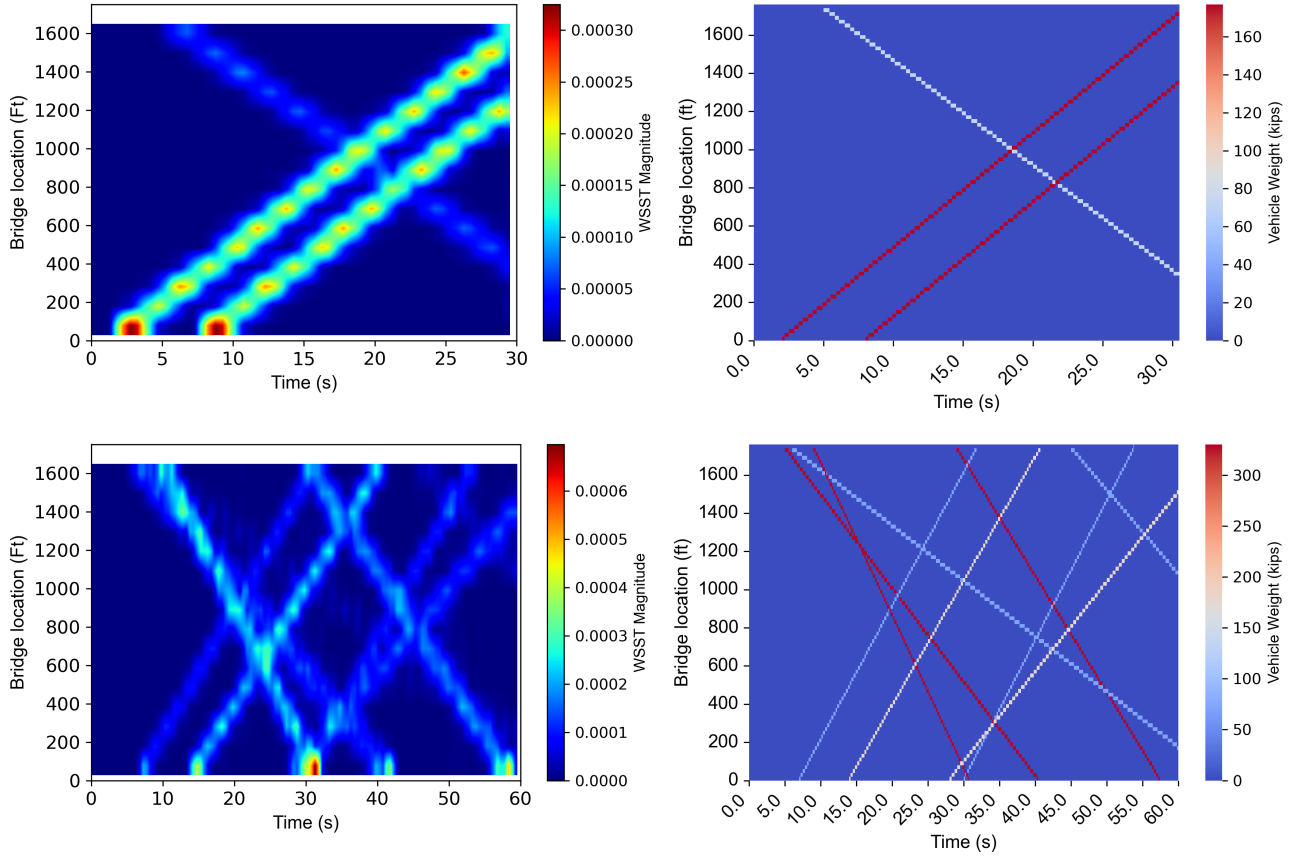


**Fig. 2** Wavelet transform coefficient on the acceleration response signals from 16 joints.



**Fig. 3** Spatio-temporal visualization of the averaged wavelet coefficient magnitudes for eight distinct frequency bins, with sensor locations mapped to their corresponding positions on the bridge structure.

The bottom two plots of Figure 4 present the results of a more complex moving load scenario, involving nine vehicles simultaneously traversing the bridge over a 60-second time horizon with speeds from 29 to 80 m/s and weights from 72 to 330 kips. This case presents a significant challenge for conventional methods in detecting moving loads due to the complexity



**Fig. 4** Vehicle trajectory maps. Top left: Identified vehicle trajectory maps using wavelet coefficients with three vehicles on the bridge, Top right: The true trajectory map of the moving load case with three vehicles on the bridge. Bottom left: Identified vehicle trajectory maps using wavelet coefficients with nine vehicles on the bridge, Bottom right: The true trajectory map of the moving load case with nine vehicles on the bridge

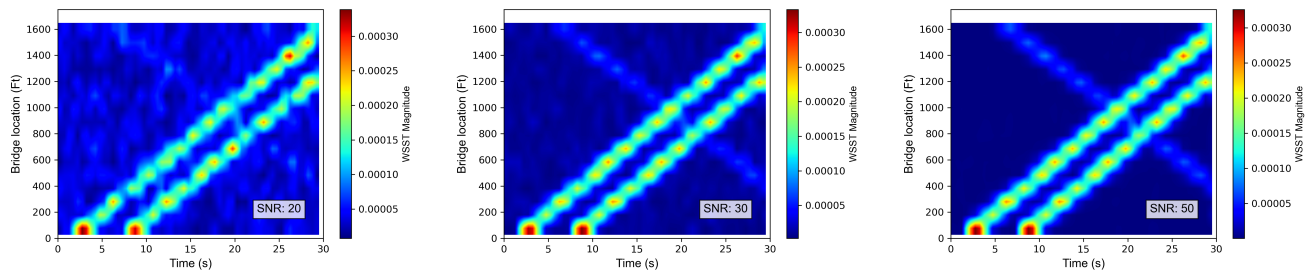
of multiple vehicles interacting with the structure. However, the proposed framework successfully identified the majority of the vehicle trajectories, even under these complex conditions. It is worth noting that in this second analysis, some vehicle trajectories appear faded, which suggests that the ability to distinguish between vehicles may be reduced when vehicles of significantly different weights are present on the bridge. This reduction in resolution is a point for future investigation, as improving the signal quality in such cases could further enhance the precision of vehicle identification.

The quality of the collected signals plays a critical role in the effectiveness of the proposed framework, particularly when dealing with high-frequency components, which are often contaminated by noise. As high-frequency content is more susceptible to interference, it becomes essential to assess the signal-to-noise ratio (SNR) and ensure that the quality of the recorded data is sufficient to distinguish meaningful structural responses from noise. Establishing the quantitative limitations of this method requires further investigation, taking into account factors such as sensor quality, the dynamic characteristics of the bridge, and the environmental conditions under which the data is collected. A systematic evaluation of these factors will determine the feasibility of applying this framework to specific bridge monitoring scenarios.

As advancements in sensor technology and signal processing techniques continue to improve, the ability to extract detailed information from acceleration data will significantly enhance. This framework has the potential to leverage these improvements, making it a promising approach for the identification of vehicle-induced loads and dynamic interactions on bridges.

Figure 5 presents the vehicle trajectory maps derived from signals with varying SNR levels of 20 dB, 30 dB, and 50 dB. The case study results demonstrate that at an SNR of 20 dB, the quality of vehicle identification decreases, with reduced clarity in the vehicle trajectories. Nevertheless, two vehicles remain clearly identifiable, indicating that the method retains some robustness even at lower signal quality. At SNR levels of 30 dB and 50 dB, the quality of identification improves significantly. At 30 dB, the identification quality is satisfactory, while at 50 dB, the method provides excellent accuracy,





**Fig. 5** The identified vehicle trajectories when the input acceleration responses are polluted to noise with three different SNR levels of 20, 30, and 50 dB.

with the vehicle trajectories being sharply defined. These findings highlight the sensitivity of the method to signal quality and underscore the importance of ensuring high SNR in field applications for optimal results. Moreover, incorporating uncertainty quantification in these assessments will allow for a more rigorous evaluation of the method's performance across different SNR levels, providing confidence in the robustness of the approach under various noise conditions.

## Conclusion

This paper presents a novel approach for bridge load identification using an array of acceleration responses collected along the bridge length. While acceleration data is traditionally employed for operational modal analysis and extracting modal characteristics, this study demonstrates that additional valuable features can be derived. The high-frequency content of the signal was extracted using a wavelet transform, providing a sharper and more detailed time-frequency representation. Through subsequent signal processing, vehicle trajectories traversing the bridge were identified with high accuracy, along with indications of vehicle weights.

Accelerometers are well-suited for this application due to their ability to capture high-frequency content, particularly when high-quality models are used. They are also cost-effective compared to other sensing technologies, making them ideal for large-scale applications. Although accelerometers are susceptible to certain types of noise, they are generally robust against electromagnetic interference, with proper noise reduction techniques and sensor placement necessary to ensure measurement accuracy. Nevertheless, further investigation is needed to assess the impact of noise on high-frequency content and its relationship with the dynamic characteristics of the bridge structure. This will help clarify the limitations of the proposed method and enhance its applicability.

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## Chapter 16

# A Surrogate Modelling Approach Based on Anti-Resonance Properties for FRF Prediction of Uncertain Dynamical Systems

Enora Denimal Goy

**Abstract** Quantifying uncertainties of dynamical responses is crucial for the design of robust mechanical structures. Computing the Frequency Response Function (FRF) is a classical tool in this context. So far, basic approaches to propagate uncertainties have led to poor prediction accuracy due to convergence issues around the peaks. Some previous works proposed numerical strategies to deal with this limitations for small systems and in specific cases. The present work proposed a new approach based on resonance and antiresonance properties to split the FRF in different sections, a surrogate can then be constructed on each section leading to better performances than classical strategies. The methodology is illustrated on a 2 dof academic system.

**Keywords** Frequency Response Function · Uncertainty propagation · Antiresonance · Gaussian process

## Introduction

Mechanical structures are subjected to numerous uncertainties impacting their dynamic response. To reach reliable prediction levels, those uncertainties must be considered when predicting the dynamical response. These uncertain parameters are usually described with a probability law and the dynamic response becomes stochastic. However, characterising this stochastic response can represent a substantial numerical cost, especially when the number of uncertainties is high or when one wants to predict the structure full frequency response function (FRF). To reduce this cost, a well-known strategy consists of constructing a surrogate model of the FRF. Straight approaches based on constructing one surrogate for the full FRF directly proved to experience convergence issues at the resonance peaks [1]. Some recent strategies proposed to cut the FRF intervals based on numerical properties to construct one surrogate per interval [2]. However, those approaches fail to deal with complex mechanical structures with more complex dynamical behaviour.

The current work proposes to exploit the resonance and antiresonance properties of mechanical structures to cut the FRF in different intervals and construct a surrogate model of the FRF on each interval. Gaussian Processes are used for the construction of the surrogate models. By splitting the FRF using physical properties, convergence issues at resonance peaks are avoid and realistic mechanical structures can be considered. The method is illustrated on an academic example. As a result, no convergence issues are observed and the approach gives promising results for future application on 3D finite element models.

## Proposed Methodology

### General objective

A linear dynamical system is described by the following equation of motion:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} \quad (1)$$

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Enora Denimal Goy  
Inria, CMAP, Ecole Polytechnique, Palaiseau, France  
e-mail: [enora.denimal-goy@inria.fr](mailto:enora.denimal-goy@inria.fr)

where  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are the mass, damping and stiffness matrices, the displacement vector, the dot denotes the derivative over time and  $\mathbf{F}$  the excitation vector of the form  $\mathbf{F}_e \cos \omega t$ . For such system, it is classical to study the Frequency Response Function (FRF), i.e. the response of the system when  $\omega$  varies on an interval. In uncertainty quantification, many studies have been devoted to the prediction and the amplitude response. The objective of the work is to propose a new and efficient strategy for this. The general workflow is explained first, and the different steps are then detailed.

### General workflow

In the following, the uncertain parameters are denoted  $\mathbf{x}$ ,  $\omega_{AR}$  denotes an antiresonance pulsation and  $\omega_R$  a resonance pulsation. The surrogate of a quantity of interest  $z$  is referred with  $\hat{z}$ .

- Step 1: for each dof  $i$  of the system, a surrogate model of each resonance and antiresonance is constructed, i.e. the different  $\hat{\omega}_{AR}^{(i)}$  and  $\hat{\omega}_R^{(i)}$  are constructed. The creation of the training set is done based on the solving of a modified eigenvalue analysis. For each dof, it defines zones between consecutive resonance and/or antiresonance.
- Step 2: a training set of the FRF is generated. It is composed of  $N$  input points  $((\omega_k, \mathbf{x}_k))_{k \in [1, N]}$  and their evaluation  $\mathbf{y}_k$ . For each dof, the training set points are grouped in the correct zone defined in Step 1.
- Step 3: for each zone and each dof, a surrogate model of the FRF is constructed.

### Antiresonance computation

The antiresonance at dof  $i$  can be obtained by resolving a modified eigenvalue problem where the  $i$ th column of the stiffness matrix is replaced by the force vector, and where the  $i$ th column of the mass matrix is replaced by a vector of 0 [3]. This strategy is employed in Step 1 to generate the training set of the antiresonance surrogates. For the resonance, a classical eigenvalue analysis

### Gaussian process regression

In the present study, Gaussian processes (GP) are employed for the creation of the surrogates. It is assumed that the data are generated by a process  $Y_i = f(X_i) + \epsilon_i$  with  $i \sim \mathcal{N}(0, \sigma^2)$ . In the current context, no noise is considered as the data comes from numerical simulations. From a training set of  $N$  points  $(X_i, Y_i)$ , a GP is constructed based on the covariance function  $\kappa$  (which depends on hyperparameters  $\theta$ ) and the mean function  $m$ . It defines the matrix  $K_{ij} = \kappa(\theta; X_i, X_j)$  and the vector  $\mu = (f(X_1), \dots, f(X_N))$ . The hyperparameters  $\theta$  are optimised to minimise the likelihood. Then, the prediction at a new points  $X_*$  is  $\mu_* = K_*^T K^{-1} \mu$  with  $K_*$  the covariance matrix between  $X_*$  and the training set. The GPs are constructed with the GPflow python package [4].

### Application on a 2-D model

#### Model presentation

The model studied is a 2dof system represented in Figure 1. The parameters are  $m = 1$  kg,  $k = 15000$  N/kg and  $c = 1$  Ns/m. In the considered case, only the stiffness is uncertain and can vary up to 10%. The frequency range of interest for the FRF prediction is  $[60, 220]$  Hz.

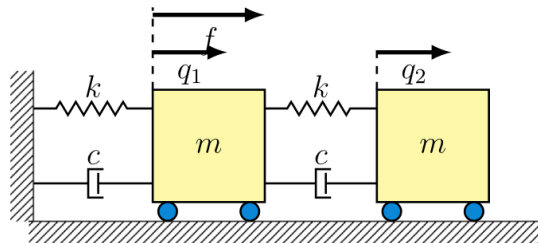
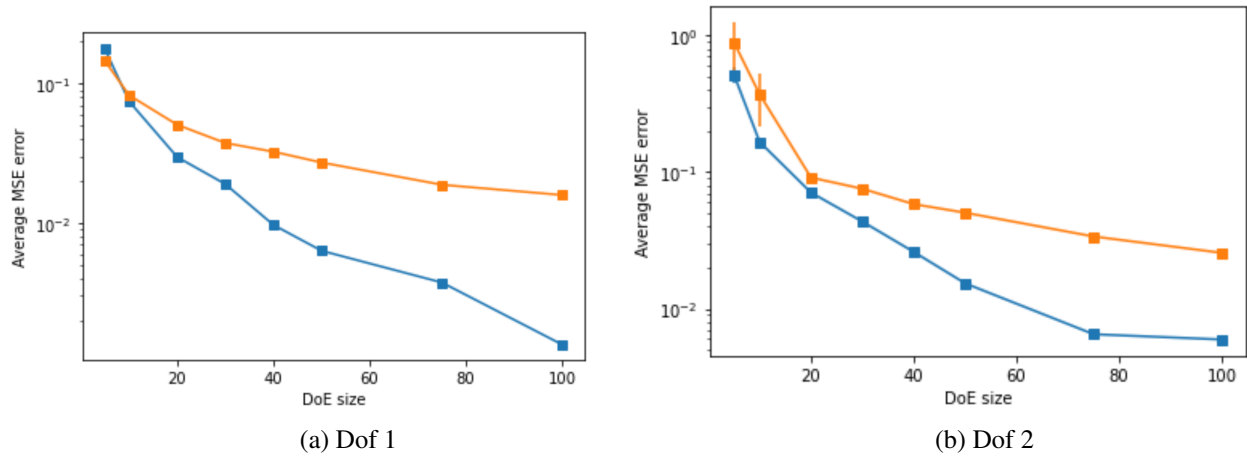


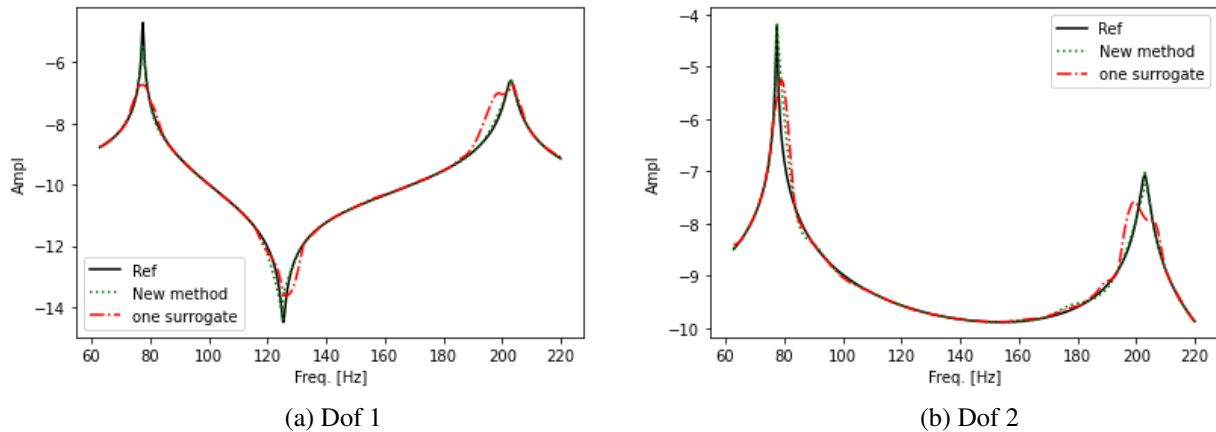
Fig. 1 Model under study

## Results

The strategy proposed previously is compared to a strategy where all points from the training set are used to create a unique surrogate of the FRF. In both cases, the data are centred before constructing the surrogate. A Matérn 5/2 kernel is taken for GPs related to the proposed strategy, and a Matérn 1/2 kernel is taken for the “unique surrogate” strategy. Concerning the present approach, for the first dof, there is one antiresonance between the two resonances, and so there are 4 zones (and 4 surrogates for the amplitude). For the second dof, there is no antiresonance over the frequency range of interest, so there are only 3 zones (and 3 surrogates for the amplitude). The quality of the surrogate is assessed by computing the Mean Squared Error (MSE) over a validation set of 100 points per zone (400 points for dof1 and 300 points for dof2). The error is computed for training set of different size, generated by LHS. For each size, 10 different training sets are generated and the average error over the 10 training sets is computed. The evolution of the error is displayed in Figure 2. The blue line corresponds to the error with the proposed strategy, whereas the orange line corresponds to the error with a “unique surrogate” strategy. The error level is one order of magnitude lower with the proposed strategy compared to the unique surrogate strategy. It shows clearly the interest in cutting the FRF into different parts between resonance and antiresonance. As an illustration, one predicted FRF is displayed in Figure 3, where the black line is the reference, the red line the results from the unique surrogate strategy and the green line from the proposed strategy. With the proposed strategy, the peaks of the FRF are well predicted, whereas oscillations are observed around the peaks with the unique surrogate strategies, leading to poor prediction quality.



**Fig. 2** Average error on the amplitude prediction for the two dofs - proposed algorithm (orange) and classic method (blue)



**Fig. 3** Prediction of one FRF - reference (black), proposed approach (green) and classic method (red)

## Conclusion

The present study has for objective to propagate uncertainty for FRF prediction of linear mechanical system. A new strategy is proposed by cutting the training set in subsets based on physical properties, namely resonance and antiresonance, and creating a surrogate for each of this subset. The computation of the antiresonance is done with a modified eigenvalue analysis. The strategy is compared to the case where a unique surrogate model is constructed. This is applied on an academic 2dof system and results proved the efficiency of the approach as lower error level and quicker convergence are observed. The results are promising for future extension to 3D structures.

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